Optimal Monetary Policy at the Zero Interest Rate Bound:
The Case of Endogenous Capital Formation*

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Abstract

This paper examines optimal monetary policy at the zero interest rate bound in a sticky price economy with variable capital. We conduct a numerical comparison between the optimal discretionary and the optimal commitment policy in response to an adverse aggregate technology shock that brings about a negative natural rate of interest. We show that, when the shock is unanticipated, the outcome under discretion is inferior to that under commitment, as shown by previous studies with fixed capital models. However, when a shock is anticipated in advance, preemptive easing by a discretionary central bank stimulates investment and accelerates capital accumulation, thereby contributing to an increase in production capacity in post-liquidity-trap periods. In this sense, the suboptimality of discretion shrinks in an economy with variable capital. Under reasonable parameter settings, this “capital channel” makes up for the central bank’s inability to make a commitment about the future course of monetary policy at the zero interest rate bound.

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1 Introduction

Recent papers on optimal monetary policy at the zero interest rate bound start their analysis by considering a situation in which the natural rate of interest falls below zero (e.g., Jung et al. 2005; Eggertsson and Woodford 2003a, b; Adam and Billi 2006, 2007; Nakov 2008). An important assumption commonly adopted in these papers is that the capital stock is fixed over time and therefore the natural rate of interest is exogenously given. This is a convenient assumption to simplify analysis, but it involves the risk that some important aspects of optimal policy responses to a negative natural rate of interest may be overlooked. In particular, it may potentially be inappropriate to ignore the consequences of monetary easing on capital accumulation as well as the effect of changes in the capital stock on the natural rate of interest. In this paper, we depart from previous studies by employing a model with variable capital, in which the natural rate of interest is endogenously determined, and examine the optimal monetary policy response to adverse shocks that bring about a decline of the natural rate of interest to a negative level.

To illustrate the role of an endogenous capital stock in an economy subject to the zero interest rate bound, let us consider a case in which a substantial decline in aggregate technology growth in period $T$ leads to a decline in the natural rate of interest to a negative level. For simplicity, it is assumed that this shock lasts only one period. The best action that a central bank can take in period $T$ is to lower the policy rate to zero. However, because of the negative natural rate of interest in period $T$, the interest rate gap, defined as the difference between the actual real interest rate and its natural rate counterpart, may take a positive value in period $T$, leading to a decline in consumption in period $T$.

How can a central bank cope with this problem? As shown by Jung et al. (2005) among others, a central bank can exploit the expectations channel to stimulate the economy through monetary policy if it can make a credible commitment about future monetary policy. Specifically, if the central bank commits to monetary easing in period $T+1$, this leads to an increase in $c_{T+1}$, thereby contributing to an increase in $c_T$, even if the interest rate gap in period $T$ remains unchanged. However, a central bank without commitment technology cannot exploit this expectations channel and has to conduct monetary policy in a discretionary manner. Since a discretionary central bank chooses to close an interest rate gap whenever possible, the consumption level in the period just after the shock, $c_{T+1}$, must coincide with its natural rate, $c^n_{T+1}$, which is defined as the level of consumption in a flexible-price economy. Note that, according to Woodford’s (2003, 2005) definition of natural rates in an economy with variable capital, $c^n_{T+1}$ depends positively on the level of capital stock at the beginning of period $T+1$; however, when capital is assumed to be fixed, $c^n_{T+1}$ is exogenously given. Therefore, a discretionary central bank has no way of avoiding a decline in consumption in period $T$.

However, there may be another channel through which the central bank can mitigate a recession due to an anticipated shock if we relax the assumption of fixed capital. Consider an economy with variable capital and
suppose that the aggregate technology shock in period $T$ is anticipated in period $T - 1$. In this case, inflation and output fall in period $T - 1$ due to the forward-looking behavior of economic agents who anticipate the adverse shock in period $T$. A discretionary central bank responds to this by lowering the policy rate in period $T - 1$, as discussed by Adam and Billi (2007). This preemptive monetary easing affects $c_T$ in two different ways. On the one hand, it stimulates investment in period $T - 1$, so that the capital stock at the end of period $T - 1$, as well as that at the end of period $T$, increases. Capital accumulation contributes to an expansion of production capacity in period $T + 1$, thereby increasing $c_{T+1}^n$. This leads to an increase in $c_{T+1}$ and finally to an increase in $c_T$. On the other hand, capital accumulation in period $T - 1$ drives down the natural rate of interest in period $T$ even further and makes the zero bound constraint bind to an even greater extent. This leads to a further decline in $c_T$. In this paper, we will conduct a numerical comparison of these two competing effects to evaluate the role of preemptive monetary easing at the zero interest rate bound in an economy with variable capital.

The main questions we will address in this paper are as follows. First, we ask whether the zero bound on the interest rate can be a binding constraint in a model with endogenous capital. Through numerical simulations, we show that, given an adverse technology shock of a certain magnitude, the natural rate of interest is less likely to fall below zero in an economy with endogenous capital than one with fixed capital. This is a direct reflection of consumption smoothing through capital adjustments. At the same time, we show that a technology shock, when it is anticipated, can potentially cause the natural rate of interest to fall below zero under reasonable parameter settings. The second question we address in this paper is how the existence of endogenous capital changes the conclusion of previous studies with fixed capital models with respect to optimal policy under discretion and commitment. We show that, when shocks are unanticipated, the outcome under discretion is inferior to that under commitment, as shown by previous studies with fixed capital models. However, when a shock is anticipated in advance, discretionary and commitment policies deliver a similar outcome. In this sense, the suboptimality of discretion shrinks in an economy with variable capital. This happens because preemptive easing by a discretionary central bank stimulates investment and accelerates capital accumulation, thereby contributing to an increase in production capacity in post-liquidity-trap periods. Under reasonable parameter settings, this “capital channel” makes up for the central bank’s inability to make a commitment about the future course of monetary policy at the zero interest rate bound.

The rest of the paper is organized as follows. Section 2 presents our model with endogenous capital formation. Specifically, it characterizes the steady state and presents the log-linearized system and the utility-based loss function. Section 3 discusses when and how frequently the zero bound constraint is binding. Section 4 characterizes the optimal commitment policy and the optimal discretionary policy, and Section 5 conducts numerical comparisons of outcomes under the two policies. Finally, Section 6 concludes the paper. Appendices A, B, and C provide technical details.
2 The model

2.1 The optimal decisions of economic agents

The analysis of this paper is based on the New Keynesian dynamic stochastic general equilibrium (DSGE) model with capital accumulation developed by Woodford (2005). There is a representative household that maximizes lifetime expected utility, 
\[ E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(C_t; \xi_t) - \int_0^1 v(h_t(i); \zeta_t) di \right\}, \]
by choosing the level of consumption, \( C_t \), the quantity of specialized labor supply, \( h_t(i) \), used to produce intermediate goods indexed by \( i \in [0, 1] \), and the quantity of state-contingent bonds. Here, \( \beta \) is the subjective discount factor and \( \xi_t \) and \( \zeta_t \) are preference shocks. Sources of household income other than through the supply of labor are dividends from firms and lump-sum transfers from government. There are two industries in this economy, each consisting of a particular type of firms. The first industry consists of a type-I firm which accumulates capital stock, while the second consists of type-II firms which hire capital and workers in competitive factor markets to produce intermediate goods that can be aggregated into private and government consumption goods as well as investment goods. The first-order conditions for the representative household are standard:

\[ \frac{v_h(h_t(i); \xi_t)}{u_c(C_t; \xi_t)} = w_t(i), \ \forall i \in [0, 1], \tag{1} \]
\[ u_c(C_t; \xi_t) Q_{t,t+1} \frac{P_{t+1}}{P_t} = \beta u_c(C_{t+1}; \xi_{t+1}), \tag{2} \]

where \( u_c(\cdot) \) is the marginal utility of consumption, \( v_h(\cdot) \) is the marginal disutility of labor, \( w_t(i) \) is the real wage for the supply of labor of type \( i \), \( Q_{t,t+1} \) is the nominal stochastic discount factor between periods \( t \) and \( t+1 \), and \( P_t \) is the price level. Equation (1) equates the marginal rate of substitution between consumption and leisure to their relative price and equation (2) is a state-by-state relationship derived from the optimal choice of state-contingent bonds. Using equation (2), the risk-free one-period gross nominal interest rate, \( R_t \), can be defined as

\[ R_t = \left\{ E_t \left[ Q_{t,t+1} \right] \right\}^{-1}. \tag{3} \]

The role of the type-I firm is to rent the existing capital stock at the beginning of period \( t \), \( K_t \), at the rental price \( \rho_t \) and make an investment decision to adjust the level of capital in the next period. It is assumed that it has to pay an adjustment cost whenever it adjusts capital. The investment expenditure function, which includes the convex adjustment cost of capital, \( I_t = I (K_{t+1}/K_t) K_t \), has the property that \( I(1) = \delta > 0, I'(1) = 1 \) and

\footnote{For tractability, we assume that capital is hired every period rather than accumulated at the firm level.}
\[ I''(1) = \epsilon_\psi > 0, \] where \( \delta \) is the depreciation rate of capital stock. The firm’s maximization problem is given by

\[
\max_{\{K_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} Q_{0,t} \frac{P_t}{P_0} \left[ \rho_t K_t - I_t \right],
\]

subject to

\[
I_t = I \left( \frac{K_{t+1}}{K_t} \right) K_t, \tag{4}
\]

The first-order condition is

\[
I' \left( \frac{K_{t+1}}{K_t} \right) = E_t Q_{t,t+1} \Pi_{t+1} \left[ \rho_{t+1} - I \left( \frac{K_{t+2}}{K_{t+1}} \right) + I' \left( \frac{K_{t+2}}{K_{t+1}} \right) \frac{K_{t+2}}{K_{t+1}} \right], \tag{5}
\]

where \( \Pi_{t+1} = P_{t+1} / P_t \).

Type-II firms are indexed by \( z \in [0, 1] \) and operate in a monopolistically competitive environment. Each firm hires workers, \( h_t(z) \), and capital, \( k_t(z) \), to produce the quantity demanded for intermediate goods by the private and the public sector, \( y_t(z) = (p_t(z)/P_t)^{-\theta} Y_t \), using constant-returns-to-scale technology, \( y_t(z) = f \left( A_t h_t(z)/k_t(z) \right) k_t(z) \).

Here, \( A_t \) is an aggregate labor-augmenting technology shock, \( f(\cdot) \) is a strictly increasing and concave function, \( y_t(z) \) is firm \( z \)'s output, \( Y_t \) is aggregate production, and \( \theta \) is the elasticity of demand for intermediate goods. Each firm faces time-dependent price stickiness \( \text{á la Calvo (1983)} \) and reoptimizes price \( p_t(z) \) whenever possible in order to maximize the discounted sum of current and expected future profits in states where the firm is unable to adjust its price again. For simplicity, it is assumed that the government imposes a subsidy of \( \tau = 1/(\theta - 1) \) per unit of production that completely removes the distortionary effect of monopolistic competition in a steady state. In this environment, the profit maximization problem of firm \( z \) is as follows:

\[
\max_{\{p_t(z)\}} E_t \sum_{k=0}^{\infty} \alpha^k Q_{t,k+1} \left[ (1 + \tau) p_t(z) y_{t+k}(z) - P_{t+k} M \left( y_{t+k}(z) \right) \right],
\]

subject to

\[
y_{t+k}(z) = \left( \frac{p_t(z)}{P_{t+k}} \right)^{-\theta} Y_{t+k}, \tag{6}
\]

where \( \alpha \) denotes the probability that the firm is not given the opportunity to reoptimize its price in each period and \( M \) is the cost function that is obtained from

\[
M \left( y_{t+k}(z) \right) = \min_{h_{t+k}(z), k_{t+k}(z)} \left\{ w_{t+k}(z) h_{t+k}(z) + \rho_{t+k} k_{t+k}(z) | y_{t+k}(z) = f \left( A_{t+k} \frac{h_{t+k}(z)}{k_{t+k}(z)} \right) k_{t+k}(z) \right\}. \tag{7}
\]
It is then straightforward to derive the first-order condition:

\[ \frac{E_t}{\rho_{t+k}} \sum_{k=0}^{\infty} (\alpha c)^k u_c (C_{t+k}; \xi_{t+k}) P_t^{\rho_{t+k} Y_{t+k} S_{t+k}(z)} = 0, \]  

(8)

where \( S_t(z) \) is the real marginal cost for firm \( z \).

Finally, the market-clearing conditions are

\[ K_t = \int_0^1 k_t(z) dz, \quad h_t(i) = h_t(z) \text{ if } i = z \]  

(9)

2.2 Log-linearized system and natural rates

In order to solve the model numerically, we log-linearize the structural equations around the zero-inflation steady state. The representative household’s Euler equation can be obtained from equations (2), (3), and (9):

\[ \left( \hat{Y}_t - \hat{I}_t - g_t \right) = E_t \left( \hat{Y}_{t+1} - \hat{I}_{t+1} - g_{t+1} \right) - \sigma \left( \hat{R}_t - E_t \hat{R}_{t+1} \right), \]  

(10)

where \( \hat{Y}_t \equiv (Y_t - Y) / Y, \hat{I}_t \equiv (I_t - I) / Y, g_t \equiv (G_t - G) / Y + \tilde{c}_t, \sigma \equiv (\sigma c Y / C)^{-1}, \sigma_c \equiv -u_{cc}(C; 0) C / u_c(C; 0) \), and \( \tilde{c}_t \) is a preference shock to the utility of consumption.\(^2\) The log-linearized optimality condition for the type-I firm is obtained from equations (5) and (9):

\[ \epsilon_\psi \left( \hat{K}_{t+1} - \hat{K}_t \right) = -\sigma^{-1} E_t \left[ \left( \hat{Y}_{t+1} - \hat{I}_{t+1} - g_{t+1} \right) - \left( \hat{Y}_t - \hat{I}_t - g_t \right) \right] 
+ \left[ 1 - \beta (1 - \delta) \right] E_t \hat{R}_{t+1} + \beta \epsilon_\psi \left( E_t \hat{K}_{t+2} - \hat{K}_{t+1} \right), \]  

(11)

\[ \hat{I}_t = k \left[ \hat{K}_{t+1} - (1 - \delta) \hat{K}_t \right]. \]  

(12)

From the optimality conditions for type-II firms, the New Keynesian Phillips curve can be derived as follows:

\[ \hat{H}_t = \kappa \hat{S}_t + \beta E_t \hat{H}_{t+1}, \]  

(13)

where \( \kappa \equiv (1 - \alpha \beta)(1 - \alpha) / (\alpha \psi), \psi \equiv 1 + \theta \nu (\rho_y - \omega) / (\rho_y - \nu), \nu \equiv v_{hh}(h; 0) h / v_h(h; 0), \text{ and } \omega \text{ and } \rho_y \text{ are the elasticity of real marginal cost and the rental price of capital with respect to a firm’s own output, respectively.} \) The expression for the average real marginal cost, \( \bar{\hat{S}}_t \equiv \int_0^1 \hat{S}_t(z) dz \), and the rental price of capital, \( \hat{R}_t \), can be obtained

\(^2\)Variables without a time subscript indicate steady-state values and those with carets indicate the percentage deviation from the steady state.
from the type-II firms’ cost-minimization problem, equation (7):

\[ \dot{S}_t = \omega \left( \dot{Y}_t - \dot{K}_t \right) + \nu \dot{K}_t + \sigma^{-1} \left( \dot{Y}_t - \dot{I}_t - g_t \right) - \omega q_t, \]  

(14)

\[ \dot{\rho}_t = \rho_q \left( \dot{Y}_t - \dot{K}_t \right) + \nu \dot{K}_t + \sigma^{-1} \left( \dot{Y}_t - \dot{I}_t - g_t \right) - \omega q_t, \]  

(15)

where \( \omega q_t \equiv \left[ (1 + \nu) a_t + h_t \right]. \) Given the path of nominal interest rates, equations (10) through (15) determine the equilibrium outcome.

In the class of DSGE models that we consider, the natural rates of endogenous variables often serve as the target for the central bank’s stabilization policy. That is, the objective of monetary policy is to make the equilibrium outcome of the actual economy as close as possible to that of a hypothetical flexible-price economy by controlling the short-term nominal interest rate. In contrast to models with a fixed capital stock, however, the target for the central bank and the nature of monetary policy in our model depend on each other. Before considering these issues further, though, two possible definitions of natural rates must be discussed. First, Neiss and Nelson (2003) propose to define natural rates as a sequence of variables

\[ \left\{ \dot{Y}_t^f, \dot{I}_t^f, \dot{K}_t^f+1^f, \dot{S}_t^f, \dot{r}_t^f \right\}_{t=0}^\infty \]  

that satisfies, in \( t = 0, 1, \ldots, \)
equations (10), (11), (12), (14), (15), as well as the mark-up pricing rule in a flexible-price economy, \( \dot{S}_t^f = 0. \) Here, \( r_t \) denotes the real interest rate. Note that the sequence of natural rates starts in period 0 in which the natural rates depend on the actual period-zero capital. However, the natural rates in subsequent periods depend on the level of capital determined in the hypothetical flexible-price economy. In a log-linearized form, the equilibrium law of motion for the natural rate of variable \( X_t \) (\( = Y_t, I_t, K_t+1, \rho_t, r_t \)) defined by Neiss and Nelson (2003) can be expressed as

\[ \dot{X}_t^f = \begin{cases} \gamma_{x,z} \dot{z}_t + \gamma_{x,k} \dot{K}_t^f & \text{for } t = 1, 2, \ldots, \\ \gamma_{x,z} \dot{z}_t + \gamma_{x,k} \dot{K}_0 & \text{for } t = 0 \end{cases} \]  

(16)

where \( \gamma_{x,z} \) and \( \gamma_{x,k} \) represent the equilibrium responses of \( X \) to a percentage deviation of the exogenous variables, \( z_t, \) and capital, respectively, from their steady state levels. On the other hand, Woodford (2003) proposes that, in equation (16), natural rates depend on the actual capital, \( \dot{K}_t, \) for all periods rather than on the natural level of capital, \( \dot{K}_t^f, \) which is determined in the hypothetical flexible-price economy. The natural rates under the two definitions diverge if the actual equilibrium capital stock is different from the flexible-price equilibrium counterpart. In our model, this occurs when a binding zero lower bound (ZLB) constraint on the nominal interest rate prevents monetary policy from guiding the actual equilibrium outcome to that under the flexible-price equilibrium.\(^5\) Unless the ZLB binds, however, our model exhibits perfect stabilization — the equilibrium outcome is equivalent to the

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\(^3\)Here, \( a_t = \log A_t \) and \( h_t \) is a preference shock to the disutility of labor.

\(^4\)Following Woodford (2003), we call the level of output in a flexible-price economy the natural rate of output. The same rule applies to consumption, investment and capital.

\(^5\)If there was a mark-up shock, the two natural rates can differ even without a binding ZLB for the nominal interest rate.
flexible-price equilibrium outcome.

2.3 Utility-based loss function

The analysis of optimal monetary policy requires a criterion on which the central bank can base its stabilization policy. Analogous to Edge (2003) and Onatski and Williams (2004), we take a second-order approximation to the utility function of the representative household around the steady state. The approximation entails an expansion of the utility from consumption and the disutility from hours worked. The derivation of the welfare criterion is given in Section A of the Appendix, which shows that the central bank’s loss function in period 0 can be expressed as the following quadratic function:

$$L_0 = E_0 \sum_{t=0}^{\infty} \beta^t \hat{L}_t + \text{t.i.p.} + O(3),$$

where

$$\hat{L}_t = \sigma^{-1} \left( \hat{Y}_t - \bar{Y}_t - g_t \right)^2 + \epsilon_\psi k \left( \hat{K}_{t+1} - \bar{K}_t \right)^2 + \omega \left( \hat{Y}_t - \bar{Y}_t \right)^2 + \phi_k^{-1} \nu \hat{K}_t^2 + 2\nu \hat{K}_t \left( \hat{Y}_t - \bar{Y}_t \right) + \kappa^{-1} \theta \hat{\Pi}_t^2 - 2\hat{Y}_t \left( \omega q_t \right) + 2 \frac{\phi_h - 1}{\phi_h} \left( \omega q_t \right) \hat{K}_t,$$

(17)

\(\phi_h\) is the elasticity of \(f^{-1}(y/k)\) with respect to \(y/k\), and t.i.p. represents terms independent of policy. Note that this loss function nests the one in the standard fixed-capital model, \(\hat{L}_t = \sigma^{-1} + \omega \hat{Y}_t^2 - 2\hat{Y}_t \left( \sigma^{-1} g_t + \omega q_t \right) + \kappa^{-1} \theta \hat{\Pi}_t^2\), as a special case when \(\epsilon_\psi \to \infty\) (hence \(\hat{K}_t \to 0\)). The loss function in our model contains additional stabilization objectives, for the following reasons. First, the variation in consumption depends on the variability of aggregate investment as well as output. While investment helps consumption smoothing, investment itself should also be smoothed out since the change in the capital stock leads to a welfare loss due to convex adjustment costs, as represented by the second term in equation (17). Second, hours worked vary as the level of capital changes, because the level of capital affects the marginal product of labor and thus the hours of work required to produce a given level of output. Time variation in the average labor supply can be decomposed into the variation in the capital-output ratio, the variation in capital itself, and the interaction of the two. Similar to the fixed-capital model, the squared inflation term in equation (17) represents the cross-sectional variation in labor supply.

As shown in the Appendix, \(L_0\) attains the global minimum if the equilibrium outcome is identical to the natural rate defined by Neiss and Nelson (2003) in every period. This implies that a central bank seeks to guide the economy as close as possible to the natural rates as defined by Neiss and Nelson (2003) if it has the technology to commit. In other words, a central bank with commitment technology ensures that stabilization targets remain consistent over time. On the other hand, a discretionary central bank changes its stabilization targets by updating
the natural rates in the way explained by Woodford (2003) whenever the actual level of capital, $K_t$, diverges from $\bar{K}_{t=0}^f$. This distinction is important to understand the nature of the optimal commitment policy and the optimal discretionary policy. While stabilization targets are completely exogenous for the optimal commitment policy, those for a discretionary central bank are endogenous in the sense that a current policy decision affects the natural rates defined by Woodford (2003) in the next period through capital, $\bar{K}_{t+1}$. Later, we will address the question of how a discretionary central bank takes into account the endogeneity of natural rates in their decision making.

3 When does the natural rate of interest fall below zero?

The main purpose of this paper is to consider the nature of optimal monetary policy when the ZLB on the nominal interest rate binds in the variable-capital DSGE model presented in Section 2.2. In our model, imperfect stabilization occurs only when the ZLB prevents the central bank from closing the real interest rate gap in some periods. As Rogoff (1998) points out, however, when the natural rate of interest is endogenously determined, there is reason to doubt the possibility of its falling below zero. In our variable-capital model, investment can help smooth consumption allocation over time, and this, in turn, diminishes the volatility of the real interest rate through the consumption Euler equation, equation (10). We argue that whether the ZLB binds in a variable-capital environment depends on the type and the size of shocks. To illustrate this, we consider a flexible-price version of the model discussed in Section 2.2 for two types of shocks – a technology shock and a government spending shock – to see if the equilibrium real interest rate (the natural rate of interest) falls below zero. The frequency of the model is quarterly and the baseline parameter values are shown in Table 1.

3.1 A technology shock

Our first finding shows that the natural rate of interest is unlikely to be negative after an unexpected technology shock which follows an AR(1) process with a persistence level that is typically assumed in the literature. Suppose the process is expressed as $a_t = \rho_a a_{t-1} + e_{a,t}$, where $\rho_a = 0.95$ and $e_{a,t}$ is a zero-mean innovation with a standard deviation equal to $\sigma_a$. Since $A_t$ is a labor-augmenting technology shock, we set the value of $\sigma_a$ in such a way that the innovation to log $A_t^\lambda$ has the same standard deviation as total factor productivity (TFP), where $\lambda$ is the elasticity of $f(x)$ with respect to $x$. Following Mankiw and Reis (2006), we set the standard deviation of the innovation to TFP to 0.0085. Together with $\lambda = \phi_h^{-1} = 0.75$ in Table 1, this implies that the value of $\sigma_a$ is 0.0113. Figure 1 shows the impulse response functions to an unexpected positive technology shock of three standard deviations. We consider a positive shock in order to obtain negative expected growth in technology under the AR(1) assumption, because this leads to a drop in the natural rate of interest as the adjustment cost of capital becomes larger. To see

\[ \text{In our model, } \lambda = \phi_h^{-1}. \]
this, it is useful to express the natural rate of interest as

$$f_{t+1}^j = \sigma^{-1} E_t \Delta \hat{C}_t \hat{K}_t$$

$$= \sigma^{-1} \left( \gamma_{c,a} E_t \Delta a_t + \eta_{c,a} \gamma_{k,a} a_t + \eta_{c,k} (\eta_{k,k} - 1) \hat{K}_t \right), \quad (18)$$

where $\gamma_{c,a}$, $\gamma_{k,a}$, and $\eta_{c,k}$ are coefficients of the optimal consumption and capital functions, $\hat{C}_t$ is the optimal consumption function, and $\hat{K}_t$ is the optimal capital function. The value of $\varepsilon_\psi$ increases, $\eta_{c,k} \to 0$ and the natural rate of interest is approximately equal to $\sigma^{-1} \gamma_{c,a} (\rho_a - 1) a_t < 0$ for $a_t > 0$. In Figure 1, we can see that the response of the natural rate of interest never falls below zero regardless of the size of the adjustment cost of capital, even though the shock is as large as three standard deviations. The reason is that the high persistence of the type of technology shock that is typically assumed in the literature gives rise to a low rate of expected growth in technology. Of course, the natural rate of interest can fall below zero if the value of $\rho_a$ is as low as 0.3 when $\varepsilon_\psi = 3$, but this is clearly not standard.

The preceding result shows that an unexpected technology shock that follows a mean reverting process such as AR(1) may not be a good candidate for explaining the underlying forces of a liquidity trap. However, equation (18) implies that, instead, a more likely candidate to account for the natural rate of interest falling below zero is a news shock that signals a decline in the expected technology level relative to the current one. Consider a perfect foresight equilibrium in which agents receive news in the initial period that a negative technology shock will hit the economy in the following period. After the shock materializes, the technology level will decay geometrically with a persistence of 0.95. In the initial period, the anticipated shock creates a sharp fall in the expected growth in technology as well as consumption. Figure 2(a) shows that the natural rate of interest will be negative for a three-standard-deviation drop in expected technology growth unless investment is extremely elastic. Note, however, that the declines become smaller as the adjustment cost of capital decreases. This implies that the negative impact that a liquidity trap might have on the economy is mitigated in the variable-capital model by the consumption smoothing of households.

To check the robustness of the results, we conduct a sensitivity analysis by varying the degree of relative risk aversion, $\sigma_c$, and the inverse of the Frisch wage elasticity of labor supply, $\nu$. Christiano (2004) points out that when $\nu = 1$ rather than 0.11 as assumed by Woodford (2003), the ZLB will not bind for the preference shocks that

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7 Here, shocks other than aggregate technology shocks are suppressed for convenience.
8 When capital is adjustable, the increase in current technology raises capital, and this will increase $E_t \hat{C}_t \hat{K}_t$ through $\eta_{c,k} \gamma_{k,a} a_t$ in equation (18). Figure 1 indicates that this effect dominates the negative technology growth effect when $\varepsilon_\psi$ becomes smaller and leads to an increasing consumption profile in some periods.
9 In the log-linearized model, the natural rate of interest is less than zero when $\hat{r}_0^p$ is $100 (\beta - 1) / \beta = -1.01\%$ away from its steady state level.
10 In Woodford (2003), $\varepsilon_\psi = 3$ is treated as a realistic value.
11 In fact, under the baseline parameter values when $\varepsilon_\psi = 3$, a two-standard-deviation shock is sufficient to obtain a negative natural rate of interest.
he assumes in his variable-capital model. The logic is that the volatility of production becomes smaller when it is more costly to vary labor supply over time, which also makes the variation in consumption smaller. While this mechanism reduces the magnitude of the decline in the natural rate of interest, it still falls below zero for a three-standard-deviation technology news shock when $\varepsilon_\phi = 3$ (Figure 2(b)). Regarding the relative risk aversion, the baseline parameter value implies a logarithmic utility function. If we assume a greater value for $\sigma_c$ than used in the baseline case, for example one between 1 and 2, which is standard in the literature, this would make consumption less volatile while requiring a greater real interest rate response, holding the consumption profile fixed. Although we do not present the results here, we found that $\sigma_c$ has no significant impact on whether the natural rate of interest falls below zero.

3.2 A government spending shock

We also examine the response of the natural rate of interest to government spending shocks. To characterize the exogenous process of government spending, we estimate an AR(1) model, $(G_t - \bar{G}_t) / \bar{Y}_t = \rho_y (G_{t-1} - \bar{G}_{t-1}) / \bar{Y}_{t-1} + e_{g,t}$, for the period 1970Q1-2009Q2, where $G_t$ is real government consumption expenditure plus real gross government investment, $\bar{Y}_t$ is real GDP, and variables with bars are HP-filtered trends with a smoothing parameter of 1,600.$^{12}$ We arrive at an estimate for $\rho_y$ of 0.75 and for the standard error of $e_{g,t}$ of 0.0018. Analogous to equation (18), the natural rate of interest can be expressed as $r_{t|0}^f = \sigma^{-1} \left( \gamma_{c,g} E_t \Delta g_{t+1} + \eta_{c,k} \gamma_{k,g} g_t + \eta_{c,k} (\eta_{k,k} - 1) \hat{K}_{t|0}^f \right)$, where $\gamma_{c,g} < 0$ and $\gamma_{k,g} > 0$. This implies that positive expected growth or an unexpected decline in government spending is required to induce a negative natural rate of interest.

In contrast to technology shocks, government spending shocks cannot cause the natural rate of interest to fall into negative territory. This is shown in Figure 3, in which the magnitude of the expected and the unexpected shock is again set to three standard deviations. Moreover, even if the size of the shocks is substantially larger, the natural rate of interest remains in positive territory. For example, it is still positive for an anticipated shock of twenty standard deviations in period 1 when $\varepsilon_\phi = 3$.\footnote{\textsuperscript{13}A shock of twenty standard deviations corresponds to an increase of approximately 3.6\% in $\bar{G}/\bar{Y} \times (G_t - \bar{G}) / \bar{G}$. Since in the United States $\bar{G}/\bar{Y} \simeq 0.19$, this means that government spending must increase by 19\% from its steady state.}

3.3 Preference shocks

The variable-capital model in our paper accommodates two types of preference shocks. The first, denoted by $\xi_t$, affects the marginal utility of consumption, while the second, denoted by $\zeta_t$, affects the marginal disutility of labor. Although it may be difficult to calibrate the exogenous processes for these preference shocks, it is possible to infer which type of preference shock is more likely to cause a liquidity trap for a given level of shock. Recall that in the

$^{12}$Data are taken from the Bureau of Economic Analysis.

$^{13}$...
log-linearized model, \( g_t \) is a linear combination of a government spending shock and the first type of preference shock, while \( \omega q_t \) is a linear combination of a technology shock and the second type of preference shock. The preceding analysis thus indicates that whereas a preference shock that affects the marginal disutility of labor may lead to a large fall in the natural rate of interest, a shock that affects the marginal utility of consumption can only generate a small decline. In this context, it is interesting to note that Christiano (2004), analyzing a preference shock that is isomorphic to the first type of preference shock, concludes that the ZLB is unlikely to bind even when there is an extremely large shock. Our result is consistent with Christiano’s (2004) and provides an additional insight into what factors can potentially cause a liquidity trap.

The analysis in this section implies that the ZLB may bind in a variable-capital economy if a negative technology shock is anticipated and its size is sufficiently large. Of course, in the case of the recent recession, in the wake of which short-term nominal interest rates are now close to zero not only in Japan but also in the United States and Europe, factors not taken into account in our model, such as the financial crisis, play a role. However, we believe that extreme pessimism with regard to the outlook for economic fundamentals may have been a factor contributing to the crisis and may have made it more difficult to deal with the liquidity trap in which these economies found themselves. The above results suggest that it is necessary to examine what an optimal monetary policy to respond to a negative natural rate of interest brought about by an anticipated technology shock looks like.

4 Optimal monetary policy at the ZLB

In this section, we analyze the optimal monetary policy for a central bank which has the technology to commit in a timelessly-optimal fashion (Woodford (2003)) and for a discretionary central bank that reoptimizes its behavior every period. We begin by characterizing the optimal commitment policy using a variable-capital version of the targeting rule that retains history dependence, as in the fixed-capital model. We then compare the optimal discretionary policy with the commitment policy as well as the discretionary policy in the fixed-capital model, with a particular focus on preemptive actions.

4.1 The optimal commitment policy

The commitment policy can be obtained by solving the Ramsey problem, which requires the minimization of the loss function, equation (17), subject to equations (10), (11), (12), (13), (14), (15), and the ZLB on the nominal interest rate, \( \hat{R}_t \geq - (\beta^{-1} - 1) \).\(^{14}\) We assume that the commitment policy is timelessly optimal, so that it can be characterized as if it has been in place from the infinite past. After substituting out \( \hat{I}_t \), \( \hat{S}_t \), and \( \hat{\rho}_t \) from the loss

\(^{14}\)The last condition, \( \hat{R}_t \geq - (\beta^{-1} - 1) \), is required for the nominal interest rate to be non-negative, since the net nominal interest rate is \( \beta^{-1} - 1 \) in the zero-inflation steady state.
function and the structural equations, we take the first-order conditions with respect to \(\hat{Y}_t\), \(\Pi_t\) and \(\hat{K}_{t+1}\):

\[
0 = (\sigma^{-1} + \omega) \hat{Y}_t - (\sigma^{-1} g_t + \omega q_t) - (\omega - \nu) \hat{K}_t - \sigma^{-1} k (\hat{K}_{t+1} - (1 - \delta) \hat{K}_t) + \phi_{1t} - \beta^{-1} \phi_{1,t-1} - \kappa (\omega + \sigma^{-1}) \phi_{2t} + \phi_{3t} + \beta^{-1} (\sigma \rho_{g} (1 - \beta (1 - \delta)) - \beta (1 - \delta)) \phi_{3,t-1}, \tag{19}
\]

\[
0 = \kappa^{-1} \theta \Pi_t - \beta^{-1} \sigma \phi_{1,t-1} + \phi_{2t} - \phi_{2,t-1}, \tag{20}
\]

\[
0 = \sigma^{-1} k^2 (\hat{K}_{t+1} - (1 - \delta) \hat{K}_t) - \beta^{-1} k^2 (1 - \delta) \left( E_t \hat{K}_{t+2} - (1 - \delta) \hat{K}_{t+1} \right) + \epsilon_k k (\hat{K}_{t+1} - \hat{K}_t) - \beta \epsilon_k k (1 - \beta (1 - \delta)) \hat{K}_{t+1} - \beta^{-1} k \hat{Y}_t + \beta \sigma^{-1} k (1 - \delta) E_t \hat{Y}_{t+1} - \beta (\omega - \nu) E_t \hat{Y}_{t+1}
\]

\[
- \beta k (1 - \delta) (\sigma^{-1} \epsilon_{g_{t+1}} + \omega \epsilon_{q_{t+1}}) + \sigma^{-1} k g_t + k \omega \epsilon_{q_{t+1}} + \beta^{-1} k \phi_{1,t-1} - k (2 - \delta) \phi_{1t} + \beta k (1 - \delta) E_t \phi_{1,t+1} + \kappa \sigma^{-1} k \phi_{2t} - \beta \kappa [\sigma^{-1} k (1 - \delta) - \omega + \nu] E_t \phi_{2,t+1} + [k (1 - \delta) + \sigma \epsilon_{\nu}] \phi_{3,t-1}
\]

\[
- \left[ k + \sigma \epsilon_{\nu} (1 + \beta) + \beta k (1 - \delta)^2 + \sigma \rho_k \{1 - \beta (1 - \delta)\} \right] \phi_{3t} + \beta [k (1 - \delta) + \sigma \epsilon_{\nu}] E_t \phi_{3,t+1}, \tag{21}
\]

\[
0 = \phi_{1t} \left( \check{R}_t + \frac{1 - \beta}{\beta} \right), \tag{22}
\]

where \(\phi_{1t}\), \(\phi_{2t}\) and \(\phi_{3t}\) are the Lagrange multipliers associated with equations (10), (13) and (12), respectively. The Kuhn-Tucker condition requires that \(\phi_{1t} > 0\) if and only if \(\check{R}_t = -(\beta^{-1} - 1)\). We show in Section B of the Appendix that \(\phi_{2t}\) and \(\phi_{3t}\) can be substituted out to express the first-order conditions in a single equation:

\[
(1 - \gamma_1 L)(1 - \gamma_3 L) \phi_{1t} = -\mu E_t \left[ \frac{1 - \mu L^{-1}}{1 - \gamma_2 L^{-1}} \left( \Pi_t + \theta^{-1} \Delta \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) \right) \right], \tag{23}
\]

where \(\Delta\) denotes the first difference and \(\mu, \gamma_1 > 1, \gamma_2 < 1, \gamma_3 < 1,\) and \(\eta < 1\) are non-negative coefficients that depend on the structural parameters. Equation (23) expresses the condition that the central bank is required to meet by appropriately controlling the nominal interest rate. This condition nests the policy criterion for the fixed-capital model, \((1 - \gamma_1 L)(1 - \gamma_3 L) \phi_{1t} = -\mu_0 \left[ A_t + \theta^{-1} \Delta \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) \right],\) since it is possible to check numerically that \(\gamma_2 = \eta\) when \(\varepsilon_\nu \rightarrow \infty\). Notice that equation (23) exhibits several similarities to the fixed-capital model of Jung et al. (2005) and Eggertsson and Woodford (2003a, b). First, policy responds to a linear combination of current inflation and the first difference of the output gap, which is defined as the deviation of actual aggregate output from its natural rate, where the natural rate is defined following Neiss and Nelson (2003). Second, the lags of \(\phi_{1t}\) represent policy inertia in the optimal commitment policy. If we interpret \(\check{R}_t + \theta^{-1} \Delta \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right)\) as augmented inflation, denoted by \(\hat{\Pi}_t\), the optimal commitment policy can be implemented through a history-dependent “inflation targeting scheme,” as explained below. A notable difference from fixed-capital models is that the optimal commitment policy responds to a linear combination of expected future augmented inflation, as we argued in Section 2.3, a central bank with commitment technology maintains the consistency of its stabilization target, which is expressed in terms of natural rates as defined by Neiss and Nelson (2003).
which is defined as $F_{2t} = E_t (1 - \eta_1 L^{-1}) / (1 - \gamma_2 L^{-1}) \hat{\Pi}_t = \sum_{j=0}^{\infty} \Psi_j E_t \hat{\Pi}_{t+j}$, where $\Psi_0 = 1$, $\Psi_1 = \gamma_2 - \eta_1$, and $\Psi_j = \gamma_2 \Psi_{j-1}$ for $j \geq 2$. The forecasts of endogenous variables matter for policy implementation due to the presence of a channel to affect future states via the capital stock.

To understand the nature of the optimal commitment policy, it is useful to express equation (23) as a history-dependent inflation targeting rule.\(^\text{16}\) Intuitively, if the ZLB binds ($\phi_{1t} > 0$) in period $t$, the subsequent policy stance will be inflationary. In order to implement such a policy, the central bank has a predetermined target for $F_{2t}$, denoted by $F_{2TAR}^{TAR}$, and attempts to set $F_{2t} = F_{2TAR}^{TAR}$ whenever possible. The only situation in which $F_{2TAR}^{TAR}$ cannot be achieved is when the ZLB prevents the central bank from providing sufficient stimulus. In this case, the central bank simply sets the overnight interest rate to zero, which gives rise to a target shortfall of $F_{2TAR}^{TAR} = F_{2t} - F_{2TAR}^{TAR} > 0$. To determine $F_{2TAR}^{TAR}$, we posit that $\phi_{1t} = \mu \Delta_t^\Pi$; that is, the target shortfall is proportionate to the Lagrange multiplier on equation (10). Substituting this relationship into equation (23), we obtain the following updating rule for the target:

$$F_{2TAR}^{TAR}_{t+1} = (\gamma_1 + \gamma_3) \Delta_t^\Pi - \gamma_1 \gamma_3 \Delta_{t-1}^\Pi.$$  \hspace{1cm} (24)

Under the assumption that $\Delta_1^\Pi = \Delta_2^\Pi = 0$ (the ZLB has never been binding before period 0), equation (24) constitutes the inflation targeting rule to implement the optimal commitment policy. Clearly, the targeting rule can be interpreted as “history-dependent inflation-forecast targeting.” For example, suppose that $\Delta_{t-1}^\Pi = 0$. If the central bank fails to achieve the target due to the zero bound constraint in period $t$, $\Delta_t^\Pi$ takes a positive value, and consequently the predetermined inflation target for the next period becomes higher than zero. Given the higher target, the central bank must adopt a looser policy stance, including a zero interest rate policy to raise inflation expectations. In sum, the optimal commitment policy in the variable-capital model fits into the history-dependent framework, whose properties are analyzed in the fixed-capital models of Eggertsson and Woodford (2003a, b) and Iwamura et al. (2006).

On the other hand, the distinction from the fixed-capital model is that the scheme involves the central bank’s forecasts of future augmented inflation. When capital is adjustable, the central bank needs to take into account future developments in augmented inflation that are caused by expected changes in the capital stock. That is, the inflation-forecast targeting in the variable-capital model contains a new channel for preemptive action against future inflation or deflation. Note that there is a subtle difference between preemptive action through this capital channel and the pre-shock behavior of a central bank discussed by Adam and Billi (2006, 2007) and Nakov (2005) among others in fixed-capital models. The preemptive action these papers focused on responds only to a decline in current augmented inflation and, more importantly, such action has no impact on future economic performance. Having said this, reactions through the existing channel of preemptive action and policy inertia are likely to be

\(^{16}\)See Eggertsson and Woodford (2003a, b) and Iwamura et al. (2006) for a detailed discussion.
more important than those through the capital channel. To see why this might be the case, we compute the relative weights on expected augmented inflation, $\Psi_j$ ($j = 1, 2, \ldots$), for different values of $\varepsilon_{\psi}$. Figure 4 shows that when $\varepsilon_{\psi} = 3$, the relative weights on the one-period and two-period ahead forecasts are merely 0.0019 and 0.0017, respectively. The figure also indicates that the weights are extremely small even for a very small value of $\varepsilon_{\psi}$. Hence, the nature of the optimal commitment policy is largely explained by the properties already known from the fixed-capital model.

4.2 The optimal discretionary policy

Unlike a central bank pursuing a commitment policy, one pursing a discretionary policy reoptimizes its policy every period, taking as given economic agents’ optimal behavior through which they form their expectations. Under a discretionary policy, economic agents will correctly anticipate that the central bank will set the nominal interest rate equal to the real interest rate as soon as the peril of a liquidity trap is gone. Since a promise to keep inflation high in the future is not credible, outcomes under a discretionary policy are suboptimal. As noted by Adam and Billi (2007), however, the inability to exploit inertial policy leads a discretionary central bank to act more preemptively than a central bank with commitment technology. Below, we focus on this aspect of discretionary policy in the variable-capital model.

The optimal discretionary policy is a time-consistent monetary policy that can be obtained by solving the following Bellman equation,

$$V_t \left( K_t, s_t \right) = \max \left\{ Y_t, \tilde{n}_t, K_{t+1} \right\} \left\{ -L_t + \beta V_{t+1} \left( \hat{K}_{t+1}, s_{t+1} \right) \right\}$$

subject to equations (10), (11), (12), (13), (14), and $\hat{R}_t \geq - (\beta^{-1} - 1)$, given $K_t$ and a vector of exogenous state variables, $s_t$. Here, the value function has a subscript $t$ because we will assume a particular deterministic path of exogenous shocks in the numerical analysis in Section 5. In this problem, $s_t$ and $\hat{K}_t$ are sufficient to characterize the solution. Although a closed-form targeting rule for the optimal discretionary policy is not obtainable in the variable-capital model, the existence of this endogenous state variable distinguishes the optimal discretionary policy in the variable-capital model from that of the fixed-capital model. That is, the optimal policy decision must take into account the marginal effect of variations in the capital stock on the behavior of economic agents.

More specifically, the capital stock can influence the central bank’s decision-making in two ways. First, periodical reoptimization involves the revision of the target paths of endogenous variables (natural rates) when the central bank is unable in the previous period to match the actual level of capital to its natural rate. This implies that

Note that a central bank with commitment technology keeps its promise to maintain the target paths of endogenous variables even if the actual capital stock deviated from the natural rate of capital as defined by Neiss and Nelson (2003).
a discretionary central bank wants to avoid large future deviations of capital from its own target path of capital. Therefore, if the future capital stock is expected to fall below the current target, the central bank has an incentive to encourage an increase in investment in advance to reduce the future decline in the capital stock.

In this context, an interesting question surrounding the optimal discretionary policy is whether the central bank has an incentive to raise the nominal interest rate to preemptively achieve a decumulation of capital stock to raise the natural rate of interest in the future. Such a policy may be considered as a preventive measure against falling into a liquidity trap. However, capital decumulation will make the future deviation of capital from the current target even larger when capital is already expected to decline below the natural rate in the future. For this reason, preemptively raising the nominal interest rate is costly in terms of stabilization.

The second way in which the capital stock can influence the central bank’s decision-making is that a preemptive easing can strengthen the productive capacity of the actual economy through an increase of capital. More capital increases the marginal product of labor and helps reduce inefficient variation in consumption and labor supply when a worsening economic outlook generates significant deflationary pressure in periods to come. As is discussed by Adam and Billi (2007), contemporaneous deflationary pressure that stems from the forward-looking behavior of economic agents is another reason why the central bank may choose to adopt a preemptive easing. If the central bank instead tightens its policy stance preemptively in an effort to raise the future natural rate of interest, it will exacerbate the downward pressure on current inflation.

In sum, preemptive easing is an important policy tool for a discretionary central bank to cope with a liquidity trap caused by an anticipated technology shock. To what extent preemptive easing is important for discretionary monetary policy in the variable-capital model compared to fixed-capital models is a quantitative issue which will be addressed in the next section. Similarly, to what extent the implementation of the optimal discretionary policy entails forecasting because of the presence of capital and the lack of commitment technology is also an interesting question which will be considered below.

5 Numerical analysis

In this section, we solve the model numerically and discuss how equilibrium outcomes may differ under the optimal commitment and the optimal discretionary policy. In addition, we examine the nature of preemptive action taken by the discretionary central bank. For simplicity, we solve for a perfect foresight equilibrium in each exercise. In light of the results in Section 3, we assume that a technology shock will lead the ZLB to bind under the parameter values given in Table 1. The value of $\varepsilon_0$ is 3 unless otherwise stated. In each exercise, the economy has been in a steady state, and, in period 0, the central bank as well as economic agents receive information that the technology level will drop in the future. The size of the shock is set to 3 standard deviations, as in Section 3. If the shock
materializes in period 1, there will be an unexpected fall in the natural rate of interest in period 0. If the shock materializes later than period 1, there will be an anticipated fall in the natural rate of interest, which the optimal monetary policy will seek to counteract preemptively. In both cases, we assume that the technology shock will last for 3 periods and technology then returns to its steady-state level.

5.1 The optimal commitment and the optimal discretionary policy

Consider an unexpected fall in the natural rate of interest in period 0, which is caused by the news that the technology level is expected to fall in period 1. Figure 5(a) shows the equilibrium paths under both types of policies. Here, real variables are expressed as the difference between the actual outcome and the natural rate level. For output and the real interest rate, it turns out that the natural rate levels based on the two different definitions are graphically indistinguishable. Therefore, the figure does not show separate lines for the two different definitions for these variables. On the other hand, the natural rate levels of capital and consumption based on Woodford’s (2003) definition and based on Neiss and Nelson’s (2003) definition clearly differ; hence, they are shown separately in the figure. The reason why the natural rate of output is insensitive to which definition is used is that the percentage deviation of capital from its steady state level is small relative to that of labor supply. However, since the level of capital stock is large, a small variation in capital could lead to a large movement in investment. This is why the two definitions of natural rates lead to a large difference in the natural rate of investment. Through the national income identity, equation (9), this translates into a difference between the natural rate of consumption based on Neiss and Nelson (2003) and that based on Woodford (2003).

To see how the revision of target paths affects equilibrium outcomes, we compare the responses under the optimal commitment policy and those under the optimal discretionary policy. In both types of policy, the nominal interest rate is lowered to the ZLB in response to a sharp decline in the natural rate of interest below zero in period 0. The inability to close the real interest rate gap in period 0 causes a recession, the depth of which depends on the policy reaction after period 1. In each period, the discretionary policy closes the real interest rate gap relative to the natural rate as defined by Woodford (2003) to achieve perfect stabilization in the sense that similarly-defined gaps for all variables are zero. However, there are persistent negative gaps for capital and consumption from their natural rates as defined by Neiss and Nelson (2003) since the discretionary central bank loses the incentive to aid economic recovery to the initially efficient paths. On the other hand, the commitment policy pushes the economy back to the targets that it has initially promised to achieve by stimulating investment in period 1, so that the level of capital stock will be closer to the natural rate of capital as defined by Neiss and Nelson (2003). As in the fixed-capital model, the history-dependent nature of this policy smoothes the fluctuation of variables around the natural rates as defined by Neiss and Nelson (2003). This difference in the target paths of real variables is unique
to the variable-capital model.

Next, let us consider an expected fall in the natural rate of interest in period 4, which follows an anticipated negative three-standard-deviation technology shock in periods 5 through 7. Since the shock is anticipated, the optimal policy can react preemptively to counter the future contraction in the production possibility frontier. From Figure 5(b), it can be seen that preemptive action under the commitment policy is fairly modest. The central bank lowers the nominal interest rate slightly below the natural rate of interest in period 3, just one period before the ZLB constraint binds. As in the fixed-capital model examined by Adam and Billi (2006), aggressive preemptive policies are relatively unimportant for a central bank with commitment technology, because policy inertia can effectively stabilize the expectations of economic agents. In contrast, for a discretionary central bank, the inability to keep promises makes it necessary to react more aggressively prior to period 4. In our example, monetary easing begins in period 1 and the nominal interest rate is decreased to the ZLB from period 3 to 4. This is consistent with the finding of Adam and Billi (2007), although the motive for preemptive action is slightly more complicated in the variable-capital model than in the fixed-capital model.

As in fixed-capital models, the deflationary pressure stemming from the forward-looking behavior of economic agents who rationally anticipate the future shock needs to be mitigated. In addition, the discretionary central bank recognizes that future central bank decisions will not necessarily be consistent with current decisions. Naturally, the discretionary central bank has an incentive to prevent future natural rates to deviate greatly from the paths that it perceives to be desirable. Specifically, before the shock, the central bank engages in monetary easing in successive steps to stimulate consumption, investment and aggregate output to levels that exceed their natural levels. These efforts not only counteract part of the deflationary pressure arising from period 4, but also push up the capital stock to exceed the natural rate level. In period 5, capital begins to fall due to the positive real interest rate gap in period 4. However, in period 5, the negative capital gap relative to the natural rate as defined by Neiss and Nelson (2003) is much smaller than that in period 1 in Figure 5(a), since capital was already accumulated in earlier periods well above the natural rate as defined by Neiss and Nelson (2003). In fact, the level of capital is close to the commitment outcome. As the level of capital in period 5 is close to the natural rate as defined by Neiss and Nelson (2003) when the preemptive policy has been implemented, it is not surprising that, after period 5, the policy outcome is closer to the commitment case in Figure 5(b) than in Figure 5(a). Clearly, the capital and consumption gaps relative to their natural rates as defined by Neiss and Nelson (2003) in Figure 5(b) are considerably smaller than in Figure 5(a), where the central bank had no time to encourage the build-up of a buffer stock of capital. These results indicate that although the outcome of the optimal discretionary policy is inferior to the commitment outcome, particularly before period 5, such a policy can nevertheless improve welfare in the sense that gaps relative to the natural rates as defined by Neiss and Nelson (2003) become smaller if the central bank has time to act preemptively.
To highlight this point further, we consider a simplistic rule of the form $\hat{R}_t = \max\left\{\hat{r}_t^n + E_t\hat{\Pi}_{t+1}, 1 - 1/\beta\right\}$. This policy controls the nominal interest rate in such a way that the real interest rate is equal to the natural rate of interest as defined by Woodford (2003) whenever possible, which is sufficient to achieve perfect stabilization after a negative natural-rate-of-interest shock. As Figure 6 shows, however, under such a simplistic rule, the nominal interest rate is raised in periods 2 and 3 to close the real interest rate gap. As a result, not only is the recession exacerbated, but the negative capital gap relative to its natural rate as defined by Neiss and Nelson (2003) also becomes enormous. Even though all gaps from the natural rates as defined by Woodford (2003) will be zero in and after period 5 under the simplistic policy, the capital and consumption gaps relative to their natural rates as defined by Neiss and Nelson (2003) are unambiguously larger than under the optimal discretionary policy. This example illustrates that an inappropriate policy response in anticipation of a liquidity trap may be detrimental to the economy even after the danger has passed. Such a policy implication for a central bank without commitment technology does not exist in fixed-capital models or even in models with inflation inertia (but without capital) where future natural rates are independent of the history of economic outcomes.\(^{18}\)

5.2 Preemptive policy and the variability of capital

How does the variability of capital affect equilibrium outcomes before and during the liquidity trap? To address this question, we compare the equilibrium outcomes under the optimal discretionary policy when $\varepsilon_\psi = 3$ and 500. In Figure 7, equilibrium outcomes are expressed in gaps relative to the natural rates as defined by Neiss and Nelson (2003). The figure shows that the inflation and consumption gaps are smaller and smoother as capital becomes more variable. That is, given the magnitude of the technology shock, the differences in the inflation and consumption paths under commitment and discretion are smaller the more variable capital becomes. The reason why the suboptimality of discretion shrinks is that, first, more variable capital leads to a smaller fall in the natural rate of interest in period 4, as we saw in Section 3. When the same technology shock translates into a smaller natural-rate-of-interest shock, the impact on inflation and the output gap will be smaller. And second, an increase in capital caused by preemptive easing contributes to increases in the production capacity in post-liquidity-trap periods, thereby mitigating at least partially the decline in inflation and output in pre-liquidity-trap periods. As a result, the optimal discretionary policy does not need to cut the nominal interest rate as far in advance as in the fixed-capital environment, as Figure 7 indicates.

\(^{18}\)In models with inflation inertia, natural rates do not depend on lagged inflation since inflation is constant in a flexible-price economy.
5.3 Is forecasting important for the optimal discretionary policy?

The optimal discretionary policy has no closed-form representation and it is not clear which variables play a critical role for the implementation of the policy. Moreover, the exposition in Section 4.2 does not answer to what extent the presence of capital and the ZLB makes the forecasting of future economic variables important when a discretionary central bank takes preemptive action. In order to address these issues, we consider a policy which attempts to satisfy the following relationship whenever possible:

\[ 0 = \left[ \tilde{\Pi}_t + \theta^{-1} \left( \hat{Y}_t - \hat{Y}_t^n \right) \right]. \quad (25) \]

When the ZLB prevents the policy from meeting this criterion, the nominal interest rate is set to zero and the right hand side of equation (25), which can be viewed as augmented inflation for the discretionary policy, is negative. The implementation criterion of the optimal discretionary policy in the fixed-capital model has the same representation as equation (25) and this static policy is sufficient to achieve perfect stabilization in the variable-capital model if the ZLB does not bind in any period. This means that the comparison of equilibrium outcomes under the optimal discretionary policy and the fixed-capital policy, equation (25), can provide us with some information on whether forecasting is critical in determining to what extent preemptive action taken by a discretionary central bank will be successful.

Figure 8 presents the differences in equilibrium outcomes when monetary policy follows the optimal discretionary rule and the fixed-capital rule. In each panel, the equilibrium outcome under the fixed-capital rule is subtracted from that of the optimal discretionary policy. Clearly, the magnitudes of the differences in each panel are extremely small. For example, the nominal interest rate differs by no more than 1 basis point except in period 2, where the difference is still only 1.6 basis points. Consequently, the impact of forecasting on outcomes in the optimal discretionary policy is quantitatively small in our model. The gains from forecasting future variables are limited since the current variables reflect a sufficient amount of information about the future recession through the forward-looking behavior of private agents. Moreover, our result suggests that, despite the introduction of investment in the model, the optimal discretionary policy can be approximated well with a criterion which only includes a linear combination of inflation and the output gap. Note that the output gap is measured by the distance from the natural rate of output following Woodford’s definition (2003) rather than from that when capital is fixed. This carries the practical implication that a discretionary central bank should make greater efforts to precisely measure current inflation and the natural rate of output than to accurately forecast their future values.
6 Conclusion

This paper examined optimal monetary policy in an economy at the zero interest rate bound with endogenous capital formation. The main findings of the paper can be summarized as follows. First, the analysis showed numerically that news about a future technology shock can potentially cause the natural rate of interest to fall below zero under reasonable parameter settings even if households can spread the impact of the shock by exploiting investment for intertemporal smoothing. Moreover, it was found that the same result is not obtained for a government spending shock. Compared to the case where capital is fixed, however, the fall in the natural rate of interest is smaller. This means that the extent of the monetary policy response will be smaller in a variable-capital environment.

Second, we derived the loss function in this economy and characterized the optimal commitment policy and the optimal discretionary policy. Our numerical exercises show that the optimal commitment policy uses inertia effectively to target the natural rates as defined by Neiss and Nelson (2003). On the other hand, the optimal discretionary policy revises target paths of variables every period if investment deviated from the natural rate in the previous period. However, if the shock to the natural rate of interest is anticipated, the optimal discretionary policy lowers the nominal interest rate in advance not only to mitigate the recession caused by the anticipated negative technology shock but also to stimulate investment and the accumulation of capital. The resulting buffer stock of capital helps reduce the deviation of capital from the natural rates as defined by Neiss and Nelson (2003) even though the discretionary central bank in each period has no incentive to adhere to the target paths determined in previous periods. In particular, preemptive policy can reduce the consumption and capital gaps measured in terms of the natural rates as defined by Neiss and Nelson (2003) relative to what they would be if the natural rate of interest had fallen unexpectedly. This implies that natural rates are an important element of monetary policy decision making when capital varies over time.

Third, for the implementation of optimal policies, the central bank’s forecasting plays a limited role. Regardless of the availability of commitment technology, the appropriate responses to current inflation and the output gap (augmented inflation) allow the central bank to attain equilibrium outcomes that are almost similar to the optimal outcomes, because the effects of an anticipated future shock are already built into current augmented inflation through economic agents’ forward-looking behavior.

Some limitations of the analysis in this paper should be highlighted. One of these is that the numerical analysis in Section 5 assumes perfect foresight. As highlighted by Adam and Billi (2007), stochastic simulations might produce quantitatively different answers to the questions we addressed in this paper. In addition, we abstracted from a richer class of propagation mechanisms such as financial frictions, which may be important for explaining the Great Depression and the financial crises that occurred in many developed economies in recent years. Addressing these issues is beyond the scope of this paper and is left for future research.
References


Table 1: Baseline parameter values for the numerical analysis

<table>
<thead>
<tr>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ Calvo parameter for price stickiness</td>
<td>0.66</td>
</tr>
<tr>
<td>$\beta$ Subjective discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_c$ Degree of relative risk aversion</td>
<td>1</td>
</tr>
<tr>
<td>$s_c$ Consumption/GDP ratio</td>
<td>0.6</td>
</tr>
<tr>
<td>$\nu$ Frisch wage elasticity of labor</td>
<td>0.11</td>
</tr>
<tr>
<td>$\phi_h$ Elasticity of $f^{-1}(y/k)$ with respect to $y/k$</td>
<td>$0.75^{-1} = 1.33$</td>
</tr>
<tr>
<td>$\omega_p$ Elasticity of $f'(f^{-1}(y/k))$ with respect to $y/k$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\theta$ Price elasticity of demand</td>
<td>$1 + 0.15^{-1} = 7.67$</td>
</tr>
<tr>
<td>$\delta$ Depreciation rate of capital</td>
<td>$0.1/4 = 0.025$</td>
</tr>
</tbody>
</table>

Notes: The values for $\alpha$ through $\theta$ are taken from Woodford (2003), Table 5.1, except for $\sigma_c = 1$, which is standard in the literature. For $\delta$, we assume a 10% annual depreciation of capital.

Figure 1: Impulse response functions to an unanticipated technology shock
Figure 2: Impulse response functions to an anticipated technology shock

(a) Baseline parameters

(b) $\nu = 1$
Figure 3: Impulse response functions to a government-spending shock

(a) Unanticipated shock

(b) Anticipated shock
Figure 4: Weights on forecasted augmented inflation
Figure 5: The optimal commitment and discretionary policies

(a) Unanticipated natural-rate-of-interest shock

(b) Anticipated natural-rate-of-interest shock
Figure 6: Simplistic policy and the optimal discretionary policy

Figure 7: The variability of capital and preemptive action
Figure 8: The difference between the optimal discretionary policy and the fixed-capital rule
Appendix A: Derivation of the loss function

The derivation of the loss function in this paper follows a similar method to the one employed by Edge (2003) and Onatski and Williams (2004). Suppose \( f(x_t; \eta_t) \) is the function to be approximated, where \( x_t \) is an endogenous variable and \( \eta_t \) is a vector of exogenous variables. Let \( x_t = g(z_t, \mu_t) \) be the function relating \( x_t \) to the vectors of endogenous variables \( z_t \) and exogenous variables \( \mu_t \). A second-order Taylor approximation to \( f(\cdot) \) with respect to \( z_t \) and \( \eta_t \) is

\[
\begin{align*}
    f(x_t; \epsilon_t) &= f(x; 0) + \left[ f_z^T, f_{\mu}^T, f_{\epsilon}^T \right] \begin{bmatrix} dz_t \\ d\mu_t \\ d\epsilon_t \end{bmatrix} + O \left( ||\eta||^3 \right), \\
    &= \left[ f_x g_z^T \Lambda(z), f_x^T \Lambda(z) + f_x g_{\mu} g_z^T \Lambda(z) + f_x g_{\epsilon} g_z^T \Lambda(z) \right] \begin{bmatrix} \hat{z}_t \\ \hat{\mu}_t \\ \hat{\epsilon}_t \end{bmatrix} + O \left( ||\epsilon||^3 \right),
\end{align*}
\]

where \( T \) is the transpose of a matrix, and \( f_z \) and \( f_{\epsilon} \) represent the first and second derivatives of \( f \) with respect to \( z \) and \( \epsilon \), respectively, evaluated at the steady state. Let \( z \) denote the steady-state value of \( z_t \) and \( \hat{z}_t \) denote the percentage deviation of \( z_t \) from \( z \). Then,

\[
dz_t = \Lambda(z) \left( \hat{z}_t + \frac{1}{2} \Lambda(\hat{z}_t) \hat{z}_t \right) + O \left( 3 \right),
\]

where \( \Lambda(z) \) is a diagonal matrix which has elements of vector \( z \) along the diagonal. Substituting this into the above equation and rearranging, we obtain

\[
\begin{align*}
    f(x_t; \epsilon_t) &= \left[ f_x g_z^T \Lambda(z), f_x^T \Lambda(z) + f_x g_{\mu} g_z^T \Lambda(z) + f_x g_{\epsilon} g_z^T \Lambda(z) \right] \begin{bmatrix} \hat{z}_t \\ \hat{\mu}_t \\ \hat{\epsilon}_t \end{bmatrix} + O \left( ||\epsilon||^3 \right),
\end{align*}
\]

To approximate the utility of consumption, let \( f = u(C_t; \xi_t), g = Y_t - I \left( \frac{K_{t+1}}{K_t} \right) K_t - G_t, z_t = [Y_t, K_{t+1}, K_t]^T \),
\[\mu_t = G_t, \epsilon_t = \xi_t, \hat{z}_t = [\bar{Y}_t, \bar{K}_{t+1}, \bar{K}_t]^T\] and \(z = [Y, K, \bar{K}]^T.\) We can compute the matrices of coefficients as follows:

\[
\Lambda(z) f_x g_z = Y u_c \begin{bmatrix}
1 \\
-k \\
(1 - \delta)k
\end{bmatrix}, \quad \Lambda(z) f_x g_z z^\top \Lambda(z) = Y u_c \begin{bmatrix}
0 & 0 & 0 \\
-\epsilon_kk & \epsilon_kk \\
0 & -\epsilon_kk & -\epsilon_kk
\end{bmatrix},
\]

\[
\Lambda(z) f_x x g_z y^\top \Lambda(z) = Y u_c \begin{bmatrix}
-\sigma^{-1} & \sigma^{-1}k & -\sigma^{-1}k(1 - \delta) \\
-\sigma^{-1}k & -\sigma^{-1}k^2 & -\sigma^{-1}k^2(1 - \delta) \\
-\sigma^{-1}k(1 - \delta) & -\sigma^{-1}k^2(1 - \delta) & -\sigma^{-1}k^2(1 - \delta)
\end{bmatrix},
\]

\[
\Lambda(z) \Lambda(f_x g_z) = Y u_c \begin{bmatrix}
1 & 0 & 0 \\
0 & -k & 0 \\
0 & 0 & (1 - \delta)k
\end{bmatrix}.
\]

If we substitute these matrices into (26), it follows that

\[
u(C_t, \xi_t) = Y u_c \left[\bar{Y}_t - k\bar{K}_{t+1} + k(1 - \delta)\bar{K}_t\right]
+ \frac{u_c Y}{2} \left[-(\sigma^{-1} - 1) \bar{Y}_t^2 - k(\sigma^{-1}k + 1 + \epsilon_k) \bar{K}_{t+1}^2 - k(\sigma^{-1}k(1 - \delta)^2 - (1 - \delta) + \epsilon_k) \bar{K}_t^2
+ 2\sigma^{-1}k\bar{Y}_t\bar{K}_{t+1} - 2\sigma^{-1}k(1 - \delta)\bar{Y}_t\bar{K}_t + 2k(\sigma^{-1}k(1 - \delta) + \epsilon_k) \bar{K}_{t+1}\bar{K}_t + 2\sigma^{-1}g_k\bar{Y}_t
- 2\sigma^{-1}k g_k \bar{K}_{t+1} + 2\sigma^{-1}k(1 - \delta)g_k \bar{K}_t\right] + \text{i.e.p.} + O(\|\xi\|^3). \quad (27)
\]

Similarly, disutility of labor can be computed by letting \(f = v(h_t(j); \xi_t), g = f^{-1}(y_t(j)/k_t(j))k_t(j)/A_t, z_t = \)

\[^{19}\text{Note that we defined } \hat{G}_t \text{ as } G_t/Y \text{ rather than in terms of the percentage deviation from its steady state value. Hence, } dG_t = Y\hat{G}_t. \text{ The formula needs to be adjusted so that the appropriate elements of the second-order coefficient matrix are multiplied by } Y \text{ or } Y^2.\]
\[
[y_t(j), k_t(j), A_t]^T \text{ and } \epsilon_t = \zeta_t. \text{ The coefficient matrices are }
\]

\[
\Lambda(z) f_x g_z = hv_h \begin{bmatrix}
\phi_h \\
(1 - \phi_h) \\
-1
\end{bmatrix}, \quad \Lambda(z) f_x g_z^\tau \Lambda(z) = hv_h \begin{bmatrix}
\omega_p \phi_h & -\omega_p \phi_h & -\phi_h \\
-\omega_p \phi_h & \omega_p \phi_h & \phi_h - 1 \\
-\phi_h & \phi_h - 1 & 2
\end{bmatrix},
\]

\[
\Lambda(z) f_x g_z g_z^T \Lambda(z) = h^2 v_{hh} \begin{bmatrix}
\phi_h^2 & \phi_h (1 - \phi_h) & -\phi_h \\
\phi_h (1 - \phi_h) & (1 - \phi_h)^2 & -(1 - \phi_h) \\
-\phi_h & -(1 - \phi_h) & 1
\end{bmatrix},
\]

\[
\Lambda(z) \Lambda(f_x g_z) = hv_h \begin{bmatrix}
\phi_h \\
0 \\
1
\end{bmatrix}.
\]

Using the fact that \(Y_u_c (C; 0) / hv_h (h; 0) = \phi_h\) in steady state,

\[
\int_0^1 v (h_t(j); \zeta_t) dj = Y_u_c \int_0^1 \left[ \hat{y}_t(j) + \frac{1 - \phi_h}{\phi_h} \hat{k}_t(j) \right] dj + \frac{Y u_c}{2} \int_0^1 \left( (1 + \omega) \hat{y}_t^2(j) + \frac{1 - \phi_h}{\phi_h} (1 - \rho_h) \hat{k}_t^2(j) - 2 (\omega - \nu) \hat{y}_t(j) \hat{k}_t(j) \right)
\]

\[
-2 \omega q_h \left( \hat{y}_t(j) + \frac{1 - \phi_h}{\phi_h} \hat{k}_t(j) \right) \right) dj + \text{t.i.p.} + O (||\zeta||^3).
\] (28)

Recall that we defined \(Y_t \equiv \left( \int_0^1 y_t(j) \frac{\phi_h}{1 + \phi_h} dj \right) \frac{\phi_h}{1 + \phi_h}\) and \(K_t \equiv \int_0^1 k_t(j) dj\). Second-order approximations to these aggregators are

\[
\hat{Y}_t = E_j \hat{y}_t(j) + \frac{1 - \theta^{-1}}{2} \text{var}_j \hat{y}_t(j) + O (3),
\]

\[
\hat{K}_t = E_j \hat{k}_t(j) + \frac{1}{2} \text{var}_j \hat{k}_t(j) + O (3).
\]

At an individual firm level, \(\hat{p}_t(j) = \omega \left( \hat{y}_t(j) - \hat{k}_t(j) \right) + \nu \hat{k}_t(j) + \sigma^{-1} \left( \hat{Y}_t - \hat{I}_t - \hat{y}_t \right) - \omega q_t\). Taking the difference between this expression and (15), we obtain \(\rho_y \left( \hat{y}_t(j) - \hat{Y}_t \right) = (\rho_y - \nu) \left( \hat{k}_t(j) - \hat{K}_t \right)\). Then,

\[
\text{var}_j \hat{k}_t(j) = \left( \frac{\rho_y}{\rho_k} \right)^2 \text{var}_j \hat{y}_t(j),
\]

\[
\text{cov}_j \{ \hat{y}_t(j), \hat{k}_t(j) \} = \frac{\rho_y}{\rho_k} \text{var}_j \hat{y}_t(j).
\]
Substituting these expressions into (28) yields

\[
\int_0^1 v(h_t(j); \zeta_t) \, dj = Y u_c \left[ \hat{Y}_t + \frac{1 - \phi_h}{\phi_h} \hat{K}_t \right] \\
+ \frac{u_c Y}{2} \left[ (1 + \omega) \hat{Y}_t^2 + \frac{1 - \phi_h}{\phi_h} (1 - \rho_k) \hat{K}_t^2 - 2 (\omega - \nu) \hat{Y}_t \hat{K}_t \\
- 2 \omega q_t \left( \hat{Y}_t + \frac{1 - \phi_h}{\phi_h} \hat{K}_t \right) + (\omega + \theta^{-1}) \text{var}_j \hat{y}_t(j) \\
+ \frac{\phi_h - 1}{\phi_h} \rho_k \text{var}_j \hat{k}_t(j) - 2 (\omega - \nu) \text{cov}_j \{ \hat{y}_t(j), \hat{k}_t(j) \} \right] \\
= Y u_c \left[ \hat{Y}_t + \frac{1 - \phi_h}{\phi_h} \hat{K}_t \right] \\
+ \frac{u_c Y}{2} \left[ (1 + \omega) \hat{Y}_t^2 + \frac{1 - \phi_h}{\phi_h} (1 - \rho_k) \hat{K}_t^2 - 2 (\omega - \nu) \hat{Y}_t \hat{K}_t \\
- 2 \omega q_t \left( \hat{Y}_t + \frac{1 - \phi_h}{\phi_h} \hat{K}_t \right) + (\omega + \theta^{-1}) \left( 1 + \frac{\rho_y - \omega}{\rho_k} \right) \text{var}_j \hat{y}_t(j) \right].
\]  
(29)

The final line in (29) used the fact that

\[
\omega + \theta^{-1} + \frac{\phi_h - 1}{\phi_h} \frac{\rho_y^2}{\rho_k} - 2 (\omega - \nu) \frac{\rho_y}{\rho_k} \\
= \omega + \theta^{-1} + \frac{\rho_y}{\rho_k} \left( \frac{\phi_h - 1}{\phi_h} \rho_y - 2 (\omega - \nu) \right) \\
= \omega + \theta^{-1} + \frac{\rho_y}{\rho_k} \left( \frac{\phi_h - 1}{\phi_h} \rho_y + \frac{\phi_h}{\phi_h - 1} \omega_p \right) - 2 (\omega - \nu) \\
= \omega + \theta^{-1} - \frac{\rho_y}{\rho_k} (\omega - \nu) \\
= \theta^{-1} \left( 1 + \frac{(\rho_y - \omega)}{\rho_k} \right) > 0.
\]

This term is positive since \( \rho_y - \omega = \phi_h \nu + \frac{\phi_h}{\phi_h - 1} \omega_p - \omega = \frac{\omega_p}{\phi_h - 1} > 0 \). The output dispersion term in (29), \( \text{var}_j \hat{y}_t(j) \), can be replaced by inflation. To show this, we log-linearize the demand function for an individual firm:

\[
\hat{y}_t(j) - \hat{Y}_t = -\theta \left( \hat{P}_t(j) - \hat{P}_t \right) + O(2).
\]

Squaring both sides and taking expectations over \( j \) then yields

\[
\text{var}_j \hat{y}_t(j) = \theta^2 \text{var}_j \hat{P}_t(j) + O(3).
\]
Following Rotemberg and Woodford (1997), the price dispersion on the right-hand side can be further transformed:

\[
\text{var}_j \hat{p}_t(j) = \text{var}_j \log p_t(j) \\
= \text{var}_j \{ \log p_t(j) - \bar{p}_{t-1} \} \quad (\text{where } \bar{p}_t \equiv E_j \{ \log p_t(j) \}) \\
= E_j \{ \log p_t(j) - \bar{p}_{t-1} \}^2 - (E_j \log p_t(j) - \bar{p}_{t-1})^2 \\
= \alpha E_j (\log p_{t-1}(j) - \bar{p}_{t-1})^2 + (1 - \alpha) (\log p^*_t - \bar{p}_{t-1})^2 - (\Delta \bar{p}_t)^2, \\
\]

(30)

where \( \log p^*_t \) is the optimal price for firms that are allowed to reoptimize at time \( t \). Note that in this model, firms are able to hire capital input for production in every period depending on their level of production in the same period. Thus, firms that are free from price rigidity must choose the same price. This also implies that

\[
E_j \log p_t(j) = \alpha E_j \log p_{t-1}(j) + (1 - \alpha) \log p^*_t.
\]

Therefore, \( \log p^*_t - \bar{p}_{t-1} = \frac{1}{1 - \alpha} \Delta \bar{p}_t \) and (30) reduces to

\[
\text{var}_j \hat{p}_t(j) = \alpha \text{var}_j \hat{p}_{t-1}(j) + \frac{\alpha}{1 - \alpha} (\Delta \bar{p}_t)^2 \\
= \alpha^{t+1} \text{var}_j \hat{p}_{t-1}(j) + \frac{\alpha}{1 - \alpha} \sum_{h=0}^{t} \alpha^h \hat{\Pi}_{t-h}^2.
\]

If we integrate this forward, we obtain:

\[
\sum_{t=0}^{\infty} \beta^t \text{var}_j \hat{p}_t(j) = \frac{\alpha}{1 - \alpha} \sum_{t=0}^{\infty} \sum_{h=0}^{t} \beta^t \alpha^h \hat{\Pi}_{t-h}^2 + \text{t.i.p.} \\
= \frac{\alpha}{(1 - \alpha)(1 - \alpha \beta)} \sum_{t=0}^{\infty} \beta^t \hat{\Pi}_{t}^2 + \text{t.i.p.}
\]

Hence, the output dispersion term can be replaced by

\[
\text{var}_j \hat{y}_t(j) = \frac{\alpha \theta^2}{(1 - \alpha)(1 - \alpha \beta)} \hat{\Pi}_{t}^2 + \text{t.i.p.} + O(3). \quad (31)
\]

As a result, (27), (29) and (31) constitute the approximation of households’ utility function.

However, for the reasons explained in Benigno and Woodford (2006), we need to show that the first-order terms in (27) and (28) cancel out. If we collect the first-order terms and use \( 1 - \phi^{-1}_h = k (\beta^{-1} - (1 - \delta)) \),

\[20\] we obtain

\[20\] This condition is obtained from the steady-state conditions for equation (5) and the first-order conditions for the cost-minimization problem in (7).
the following relationship:

\[
\hat{\dot{Y}}_t - k\dot{K}_{t+1} + k (1 - \delta) \dot{K}_t - \left[ \hat{\dot{Y}}_t + \frac{1}{\phi_h} \dot{K}_t \right] = -k \dot{K}_{t+1} + k (1 - \delta) \dot{K}_t - \frac{1}{\phi_h} \dot{K}_t
\]

Integrating (32) forward, we obtain

\[
-\sum_{t=0}^{\infty} \beta^t k \left[ \dot{K}_{t+1} - \beta^{-1} \dot{K}_t \right] = \frac{K}{\beta} \dot{K}_0 = \text{t.i.p.}
\]

Thus, the welfare function consists of second-order terms only. Collecting the remaining terms from (27), (29) and (31), the loss function of the central bank can be expressed as \( L_0 = \frac{Y_{m}}{2} \hat{L}_0 + \text{t.i.p.} + O(3) \), where \( \hat{L}_0 \) is given by equation (17) in the text.

Finally, we derive the loss function expressed in gaps between the actual levels of real variables and their natural rates as defined by Neiss and Nelson (2003). This requires replacing the exogenous terms that appear in the loss function with the natural rates as defined by Neiss and Nelson (2003). Using the previously mentioned steady-state relationship, \( 1 - \phi_h^{-1} = k (\beta^{-1} - (1 - \delta)) \), and \( \rho_k \equiv \rho_k - \nu = (\phi_h - 1) \nu + \phi_h \omega_p / (\phi_h - 1) \), the loss function, (17), can be expressed as follows:

\[
\hat{L}_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \left( \sigma^{-1} + \omega \right) \hat{Y}_t \hat{Y}_t \sigma^{-1} \hat{I}_t \right] + \left( \rho_k k (\beta^{-1} - (1 - \delta)) \right) \hat{K}_t^2 - 2 \sigma^{-1} \hat{Y}_t \hat{I}_t
\]

Furthermore, the natural rates as defined by Neiss and Nelson (2003) satisfy the real marginal cost function, (14), and the Euler equation for investment, (11):

\[
\sigma^{-1} g_t + \omega q_t = (\omega + \sigma^{-1}) \left( \hat{Y}_t \hat{I}_t \sigma^{-1} \hat{I}_t \right) - (\omega - \nu) \dot{K}_t \]

We substitute the term \( \sigma^{-1} g_{t+1} + \omega q_{t+1} \) in equation (36) using equation (35) and then multiply both sides of the
The next step then is to substitute out of the loss function using equation (35):

\[
\sigma^{-1} g_t \hat{K}_{t+1} = \sigma^{-1} E_t g_{t+1} \hat{K}_{t+1} + \hat{K}_{t+1} \left[ \sigma^{-1} \hat{Y}^f_{t|0} - \sigma^{-1} \hat{I}^f_{t|0} - \epsilon_\psi \left( \hat{K}^f_{t+1|0} - \hat{K}^f_{t|0} \right) + \beta \epsilon_\psi \left( E_t \hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0} \right) \right] + \{(1 - \beta (1 - \delta)) (\rho_y - \omega - \sigma^{-1}) - \sigma^{-1} \beta (1 - \delta)\} E_t \hat{Y}^f_{t+1|0} + \sigma^{-1} E_t \hat{I}^f_{t+1|0} + \{(1 - \beta (1 - \delta)) (\omega - \nu - \rho_k) \hat{K}^f_{t+1|0} \}.
\]

(37)

Given equations (35) and (37), we express the cross-product terms in the third line of the loss function, (34), in natural rates. To do this, we substitute equation (37) into \( \sigma^{-1} g_t \hat{K}_{t+1} + \beta^{-1} \omega q_t \hat{K}_t \) in the third line of (34):

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ \sigma^{-1} g_t \hat{K}_{t+1} + \beta^{-1} \omega q_t \hat{K}^f_{t+1|0} \right] = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \sigma^{-1} g_t + \omega q_t \right] \hat{K}_t + E_0 \sum_{t=0}^{\infty} \beta^t \hat{K}_{t+1} \left[ \sigma^{-1} \hat{Y}^f_{t|0} - \sigma^{-1} \hat{I}^f_{t|0} \right] - \epsilon_\psi \left( \hat{K}^f_{t+1|0} - \hat{K}^f_{t|0} \right) + \beta \epsilon_\psi \left( E_t \hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0} \right) + \{(1 - \beta (1 - \delta)) (\rho_y - \omega - \sigma^{-1}) - \sigma^{-1} \beta (1 - \delta)\} E_t \hat{Y}^f_{t+1|0} + \sigma^{-1} E_t \hat{I}^f_{t+1|0} + \{(1 - \beta (1 - \delta)) (\omega - \nu - \rho_k) \hat{K}^f_{t+1|0} \}.
\]

(38)

The next step then is to substitute out \( \sigma^{-1} g_t + \omega q_t \) from equation (38) and the first two brackets in the third line of the loss function using equation (35):

3rd line of \( \hat{L}_0 \)

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ \hat{Y}_t + k \left\{ \beta^{-1} (1 - \delta) \right\} \hat{K}_t \right] \left( \sigma^{-1} g_t + \omega q_t \right) + 2k E_0 \sum_{t=0}^{\infty} \beta^t \hat{K}_{t+1} \left[ \sigma^{-1} \hat{Y}^f_{t|0} - \sigma^{-1} \hat{I}^f_{t|0} - \epsilon_\psi \left( \hat{K}^f_{t+1|0} - \hat{K}^f_{t|0} \right) + \beta \epsilon_\psi \left( E_t \hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0} \right) \right] + \{(1 - \beta (1 - \delta)) (\rho_y - \omega - \sigma^{-1}) - \sigma^{-1} \beta (1 - \delta)\} \hat{Y}^f_{t+1|0} + \sigma^{-1} \hat{I}^f_{t+1|0} + \{(1 - \beta (1 - \delta)) (\omega - \nu - \rho_k) \hat{K}^f_{t+1|0} \} = -2 E_0 \sum_{t=0}^{\infty} \beta^t \left[ (\omega + \sigma^{-1}) \hat{Y}^f_{t|0} \hat{Y}_t - \sigma^{-1} \hat{I}^f_{t|0} \hat{Y}_t - (\omega - \nu) \hat{K}^f_{t|0} \hat{K}_t - k \left\{ \beta^{-1} (1 - \delta) \right\} (\omega + \sigma^{-1}) \hat{Y}^f_{t|0} \hat{K}_t + \sigma^{-1} k \left\{ \beta^{-1} (1 - \delta) \right\} \hat{I}^f_{t|0} \hat{K}_t + k \left\{ \beta^{-1} (1 - \delta) \right\} (\omega - \nu) \hat{K}^f_{t|0} \hat{K}_t - \sigma^{-1} k \hat{Y}^f_{t|0} \hat{K}_{t+1} + \sigma^{-1} k \hat{I}^f_{t|0} \hat{K}_{t+1} + \epsilon_\psi k \left( \hat{K}^f_{t+1|0} - \hat{K}^f_{t|0} \right) \hat{K}_{t+1} - \beta \epsilon_\psi k \left( \hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0} \right) \hat{K}_{t+1} - k \left\{ \left(1 - \beta (1 - \delta)\right) (\rho_y - \omega - \sigma^{-1}) - \sigma^{-1} \beta (1 - \delta) \right\} \hat{Y}^f_{t+1|0} \hat{K}_{t+1} - \sigma^{-1} k \hat{I}^f_{t+1|0} \hat{K}_{t+1} - k \left\{ \left(1 - \beta (1 - \delta)\right) (\omega - \nu - \rho_k) \hat{K}^f_{t+1|0} \hat{K}_{t+1} \right\}. \]

37
We collect and simplify the coefficients on $\hat{Y}_{t|0}^f\hat{K}_t$ and $\hat{K}_{t|0}^f\hat{K}_t$ as follows:

$$
\begin{align*}
\left[\hat{Y}_{t|0}^f\hat{K}_t\right] &:= -\beta^{-1}k \left\{ \{1 - \beta (1 - \delta)\} (\omega + \sigma^{-1}) + (1 - \beta (1 - \delta)) \left( \rho_y - \omega - \sigma^{-1} \right) - \sigma^{-1} \beta (1 - \delta) \right\} \\
&= -\beta^{-1}k \left\{ (1 - \beta (1 - \delta)) \rho_y - \sigma^{-1} \beta (1 - \delta) \right\} \\
&= \sigma^{-1} (1 - \delta) - \frac{\phi_h}{\phi_h} \left( \phi_h \nu + \frac{\phi_h}{\phi_h - 1} \omega_p \right) \\
&= \sigma^{-1} (1 - \delta) - (\omega - \nu), \\
\left[\hat{K}_{t|0}^f\hat{K}_t\right] &:= -\beta^{-1}k \left\{ \{1 - \beta (1 - \delta)\} (\omega - \nu) - (1 - \beta (1 - \delta)) (\omega - \nu - \rho_k) \right\} \\
&= \beta^{-1}k \left\{ 1 - \beta (1 - \delta) \right\} \rho_k.
\end{align*}
$$

Then,

\begin{align*}
3rd \text{ line of } \hat{L}_0 \\
&= -2E_0 \sum_{t=0}^{\infty} \beta^t \left[ \left( \omega + \sigma^{-1} \right) \hat{Y}_{t|0}^f\hat{Y}_t - \sigma^{-1} \hat{I}_{t|0}^f\hat{Y}_t - (\omega - \nu) \hat{K}_{t|0}^f\hat{Y}_t - (\omega - \nu) \hat{Y}_{t|0}^f\hat{K}_t \\
&\quad - \sigma^{-1} \hat{Y}_{t|0}^f k \left\{ \hat{K}_{t+1} - (1 - \delta) \hat{K}_t \right\} + \beta^{-1}k \left\{ 1 - \beta (1 - \delta) \right\} \rho_k \hat{K}_{t|0}^f\hat{K}_t + \sigma^{-1}k \hat{I}_{t|0}^f \left\{ \hat{K}_{t+1} - (1 - \delta) \hat{K}_t \right\} \\
&\quad + \varepsilon \omega k \left( \hat{K}_{t+1|0}^f - \hat{K}_{t|0}^f \right) \hat{K}_{t+1} - \varepsilon \omega k \left( \hat{K}_{t+1|0}^f - \hat{K}_{t|0}^f \right) \hat{K}_t \right]. 
\end{align*}

Finally, we use the result (39) in (34) to obtain the loss function expressed in terms of the gaps between the levels of real variables and their natural rates as defined by Neiss and Nelson (2003):

$$
\begin{align*}
\hat{L}_0 = \sum_{t=0}^{\infty} \beta^t \left[ (\sigma^{-1} + \omega) \left( \hat{Y}_{t|0}^2 - 2\hat{Y}_{t|0}^f\hat{Y}_t \right) + \sigma^{-1} \left( \hat{I}_{t|0}^2 - 2\hat{I}_{t|0}^f\hat{I}_t \right) + \rho_k k \left( \beta^{-1} - (1 - \delta) \right) \left( \hat{K}_{t|0}^2 - 2\hat{K}_{t|0}^f\hat{K}_t \right) \\
+ \varepsilon \omega k \left( \hat{K}_{t+1|0}^2 - 2\hat{K}_{t+1} + \hat{K}_t^2 \right) + \hat{K}_{t|0}^2 \hat{K}_{t+1|0|0} + 2\hat{K}_{t|0}^f \hat{K}_{t+1|0} + 2\hat{K}_{t|0} \hat{K}_{t+1} + \hat{K}_{t|0}^f \hat{K}_t - 2\hat{K}_{t|0}^f \hat{K}_t \\
- 2\sigma^{-1} \left( \hat{Y}_{t|0}^f\hat{I}_t - \hat{Y}_{t|0}^f\hat{I}_{t|0} - 2(\omega - \nu) \left( \hat{Y}_{t|0}^f\hat{K}_t - \hat{K}_{t|0}^f\hat{Y}_t \right) + \kappa^{-1} \theta \hat{I}_t^2 \right) \\
= \sum_{t=0}^{\infty} \beta^t \left[ (\sigma^{-1} + \omega) \left( \hat{Y}_{t|0}^2 - 2\hat{Y}_{t|0}^f\hat{Y}_t \right) + \sigma^{-1} \left( \hat{I}_{t|0}^2 - 2\hat{I}_{t|0}^f\hat{I}_t \right) + \rho_k k \left( \beta^{-1} - (1 - \delta) \right) \left( \hat{K}_{t|0}^2 - 2\hat{K}_{t|0}^f\hat{K}_t \right) \\
+ \varepsilon \omega k \left( \hat{K}_{t+1|0}^2 - 2\hat{K}_{t+1} + \hat{K}_t^2 \right) + \hat{K}_{t|0}^2 \hat{K}_{t+1|0|0} + 2\hat{K}_{t|0}^f \hat{K}_{t+1|0} + 2\hat{K}_{t|0} \hat{K}_{t+1} + \hat{K}_{t|0}^f \hat{K}_t - 2\hat{K}_{t|0}^f \hat{K}_t \\
- 2(\omega - \nu) \left( \hat{Y}_{t|0}^f\hat{I}_t - \hat{Y}_{t|0}^f\hat{I}_{t|0} - 2(\omega - \nu) \left( \hat{Y}_{t|0}^f\hat{K}_t - \hat{K}_{t|0}^f\hat{Y}_t \right) + \kappa^{-1} \theta \hat{I}_t^2 \right) + \text{t.i.p.} \right]. 
\end{align*}
$$

The above result, (40), implies that the loss function is minimized if inflation, the output gap, and the capital
gap are zero in every period. The structural equations (10) through (15) can also be expressed in gap form:

\[
\begin{align*}
\left( \hat{Y}_t - \hat{Y}_t^f \right) - \left( \hat{I}_t - \hat{I}_t^f \right) &= E_t \left[ \left( \hat{Y}_{t+1} - \hat{Y}_{t+1}^f \right) - \left( \hat{I}_{t+1} - \hat{I}_{t+1}^f \right) \right] - \sigma \left( \hat{R}_t - E_t \Delta \eta_{t+1} - \hat{r}_{t|0}^f \right), \\
&+ \left[ 1 - \beta (1 - \delta) \right] E_t \left( \hat{\rho}_{t+1} - \hat{\rho}_{t+1}^f \right) \\
&+ \beta \epsilon \left[ E_t \left( \hat{K}_{t+2} - \hat{K}_{t+2|0}^f \right) - \left( \hat{K}_{t+1} - \hat{K}_{t+1|0}^f \right) \right], \\
&+ \beta \epsilon \left[ E_t \left( \hat{K}_{t+2} - \hat{K}_{t+2|0}^f \right) - \left( \hat{K}_{t+1} - \hat{K}_{t+1|0}^f \right) \right], \\
\left( \hat{I}_t - \hat{I}_t^f \right) &= k \left[ \left( \hat{K}_{t+1} - \hat{K}_{t+1|0}^f \right) - (1 - \delta) \left( \hat{K}_{t} - \hat{K}_{t|0}^f \right) \right], \\
\Delta \eta_t &= \kappa \left( \hat{S}_t - \hat{S}_{t|0}^f \right) + \beta E_t \Delta \eta_{t+1}, \\
\left( \hat{S}_t - \hat{S}_{t|0}^f \right) &= \omega \left[ \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) - \left( \hat{K}_t - \hat{K}_{t|0}^f \right) \right] + \nu \left( \hat{K}_t - \hat{K}_{t|0}^f \right) \\
&+ \sigma^{-1} \left[ \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) - \left( \hat{I}_t - \hat{I}_{t|0}^f \right) \right], \\
\left( \hat{\rho}_t - \hat{\rho}_{t|0}^f \right) &= \rho_y \left[ \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) - \left( \hat{K}_t - \hat{K}_{t|0}^f \right) \right] + \nu \left( \hat{K}_t - \hat{K}_{t|0}^f \right) \\
&+ \sigma^{-1} \left[ \left( \hat{Y}_t - \hat{Y}_{t|0}^f \right) - \left( \hat{I}_t - \hat{I}_{t|0}^f \right) \right].
\end{align*}
\]

The equations clearly show that if the output and the capital gap are zero in every period, then inflation is also zero in every period. The equations also show that perfect stabilization is obtained if the central bank is able to set the nominal interest rate equal to the natural rate of interest as defined by Neiss and Nelson (2003) in every period. In our model, this is possible if the ZLB does not bind in any period.

---

21 If the capital gap is zero in every period, then the investment gap, $\hat{I}_t - \hat{I}_{t|0}^f$, is zero every period.
Appendix B: Derivation of equation (23)

In the text, equation (23) is used to simplify the representation of the optimal commitment policy. Here, we show how this condition is derived. Since we assume that the central bank makes a commitment from a timeless perspective, the policy is implemented as if conditions (19) through (21) hold from the infinite past. In gap form, these conditions can be expressed as follows:\[22\]

\[
A_1 \ddot{Y}_t + (A_2 + A_3 L) \dddot{K}_{t+1} + (A_4 + A_5 L) \phi_{1t} + A_6 \phi_{2t} + (A_7 + A_8 L) \phi_{3t} = 0, \tag{47}
\]

\[
B_1 \ddot{X}_t + B_2 L \phi_{1t} + (B_3 + B_4 L) \phi_{2t} = 0, \tag{48}
\]

\[
(C_1 + C_2 L^{-1}) \ddot{Y}_t + (C_3 L^{-1} + C_4 + C_5 L) \dddot{K}_{t+1} + (C_6 L^{-1} + C_7 + C_8 L) \phi_{1t}
+ (C_9 L^{-1} + C_{10}) \phi_{2t} + (C_{11} L^{-1} + C_{12} + C_{13} L) \phi_{3t} = 0, \tag{49}
\]

where \(\dddot{X}_t = \dddot{X}_{t+1}, L^{-1}\) and \(L\) are lead and lag operators, respectively, and each coefficient is given below:

\[
A_1 = \sigma^{-1} + \omega, \quad A_2 = -\sigma^{-1} k, \quad A_3 = \sigma^{-1} k (1 - \delta) - (\omega - \nu), \quad A_4 = 1,
A_5 = -\beta^{-1}, \quad A_6 = -\kappa (\sigma^{-1} + \omega), \quad A_7 = 1, \quad A_8 = \beta^{-1} [\sigma \rho_k (1 - \beta (1 - \delta)) - \beta (1 - \delta)],
B_1 = \kappa^{-1} \theta, \quad B_2 = -\beta^{-1} \sigma, \quad B_3 = 1, \quad B_4 = -1,
C_1 = \sigma^{-1} k, \quad C_2 = -\beta \sigma^{-1} k (1 - \delta) + \beta (\omega - \nu), \quad C_3 = \beta \sigma^{-1} k^2 (1 - \delta) + \beta \varepsilon \sigma k,
C_4 = -\sigma^{-1} k^2 (1 + \beta) (1 - \delta)^2 + \varepsilon \sigma k (1 + \beta) + \rho_k k [1 - \beta (1 - \delta)],
C_5 = \sigma^{-1} k^2 (1 - \delta) + \varepsilon \sigma k, \quad C_6 = -\beta k (1 - \delta), \quad C_7 = k (2 - \delta), \quad C_8 = -k \beta^{-1},
C_9 = \beta \kappa [\sigma^{-1} k (1 - \delta) - (\omega - \nu)], \quad C_{10} = -\kappa \sigma^{-1} k, \quad C_{11} = -\beta [k (1 - \delta) + \varepsilon \sigma],
C_{12} = (k + \varepsilon \sigma) (1 + \beta) \sigma + \beta k (1 - \delta)^2 + \rho_k k [1 - \beta (1 - \delta)], \quad C_{13} = -k (1 - \delta) - \varepsilon \sigma.
\]

We will substitute out \(\phi_{2t}\) and \(\phi_{3t}\) from equations (47) through (49) as follows. First, we eliminate \(\phi_{3t}\) from equations (47) and (49) by computing \((C_{11} L^{-1} + C_{12} + C_{13} L) \times (47) - (A_7 + A_8 L) \times (49)\) to obtain

\[
(D_1 L^{-1} + D_2 + D_3 L) \ddot{Y}_t + (D_4 L^{-1} + D_5 + D_6 L) \dddot{K}_{t+1} + (D_7 L^{-1} + D_8 + D_9 L) \phi_{1t}
+ (D_{10} L^{-1} + D_{11} + D_{12} L) \phi_{2t} = 0. \tag{50}
\]

\[^{22}\text{These conditions can also be obtained by minimizing the loss function (40) subject to equations (41) through (46) and the ZLB.}\]
Then, we eliminate $\phi_{2t}$ by computing $(D_{10}L^{-1} + D_{11} + D_{12}L) \times (48) - (B_3 + B_4L) \times (50)$. This yields

$$
\begin{aligned}
\theta \left( E_1L^{-1} + E_2 + E_3L \right) \tilde{\Pi}_t + \left( E_4L^{-1} + E_5 + E_6L \right) (1 - L) \tilde{Y}_t + \left( E_7L^{-1} + E_8 + E_9L \right) \tilde{K}_{t+1} + (E_{10}L^{-1} + E_{11} + E_{12}L + E_{13}L^2 + E_{14}L^3) \phi_{1t} = 0.
\end{aligned}
$$

(51)

After tedious algebra, we can verify that $E_{4} = E_{1}, E_{5} = E_{2}, E_{6} = E_{3}, E_{7} = E_{8} = E_{9} = 0$. This means that if the natural rate of interest is always positive ($\phi_{1t} = 0$ for all $t$), then it reduces to the familiar criterion for the commitment policy:

$$
\tilde{\Pi}_t + \theta^{-1} \Delta \tilde{Y}_t = 0.
$$

In our model, $\phi_{1t} > 0$ for some $t$, and hence

$$
\begin{aligned}
(E_{10}L^{-1} + E_{11} + E_{12}L + E_{13}L^2 + E_{14}L^3) \phi_{1t} = -\theta \left( E_1L^{-1} + E_2 + E_3L \right) \left[ \tilde{\Pi}_t + \theta^{-1} \Delta \tilde{Y}_t \right].
\end{aligned}
$$

Then, the values of $\gamma_1 > 1, \gamma_2 < 1, \gamma_3 < 1, \gamma_4 < 1, \eta < 1, \xi < 1$ such that $E_{10}L^{-1} + E_{11} + E_{12}L + E_{13}L^2 + E_{14}L^3 = (1 - \gamma_1 L) (1 - \gamma_2 L^{-1}) (1 - \gamma_3 L) (1 - \gamma_4 L)$ and $E_1L^{-1} + E_2 + E_3L = (1 - \eta L) (1 - \xi L)$ can be found numerically. Since $\gamma_4 = \xi$ numerically, the above expression can be written as

$$
(1 - \gamma_1 L)(1 - \gamma_3 L) \phi_{1t} = -\mu_0 \left( \frac{1 - \eta L}{1 - \gamma_2 L^{-1}} \right) \left[ \tilde{\Pi}_t + \theta^{-1} \Delta \tilde{Y}_t \right].
$$

41
Appendix C: Numerical method to compute the optimal discretionary solution

In this Appendix, we present the method of computing the optimal discretionary solution in Section 5, where the technology shock is assumed to follow a particular deterministic process. Given this kind of technology shock, the optimal discretionary solution can be computed by solving the problem backwards from period 8 when the technology returns to its steady-state level.

More specifically, we first compute the value and policy functions under the optimal discretionary policy for period 8. Given the nature of the technology shock after period 8, this problem can be solved in the linear-quadratic framework using the algorithm developed by Söderlind (1999). Then, for periods 0 to 7, we consider all possible combinations regarding the periods in which the central bank sets the nominal interest rate to zero. For example, if the nominal interest rate is set to zero only in period 4, using a binomial expression, this situation can be represented by \([0, 0, 0, 1, 0, 0, 0]\). The total number of possible combinations is \(2^8\). For each case, we compute the value and policy functions for each period that satisfy the initial condition on the capital stock, \(\hat{K}_0 = 0\). Finally, we eliminate cases where either the ZLB constraint is violated or the central bank has an incentive to deviate from the assumed zero interest rate in any period given the expectation of economic agents. The details of the computational procedure are explained below.

Step 1: Dynamic programming specification

We write the Bellman equation for the problem in periods 0 to 7 as follows:

\[
\begin{align*}
\left[ \hat{K}_t, s_t \right] P_t \begin{bmatrix} \hat{K}_t \end{bmatrix} s_t &= \min_{(x_t)} \left\{ \begin{bmatrix} x_t^T \hat{K}_t, \omega q_t \end{bmatrix} Q \begin{bmatrix} x_t \end{bmatrix} + \beta \begin{bmatrix} \hat{K}_{t+1}, s_{t+1} \end{bmatrix} P_{{t+1}} \begin{bmatrix} \hat{K}_{t+1} \end{bmatrix} s_{t+1} \right\} \quad (52)
\end{align*}
\]

s.t.

\[
\begin{align*}
M_1 x_{t+1} + M_2 x_t + M_3 \hat{K}_t + M_4 q_t + M_5 (1 - 1/\beta) &= 0 \quad \text{if} \quad \hat{R}_t = 1 - 1/\beta, \\
N_1 x_{t+1} + N_2 x_t + N_3 \hat{K}_t + M_4 q_t &= 0 \quad \text{if} \quad \hat{R}_t > 1 - 1/\beta
\end{align*}
\]

where

\[
\begin{align*}
x_t &= \left[ \Pi_t, \hat{y}_t, \hat{K}_{t+1} \right]^T, \quad q_t = \left[ \omega q_t, \omega q_{t+1} \right]^T, \quad s_t = \left[ \omega q_t, \omega q_{t+1}, \ldots, \omega q_7, (1 - 1/\beta) \right]^T. 
\end{align*}
\]

\(Q\) and \(P\) represent the coefficients in the loss function, (17), and the value function in quadratic form, respectively. In periods where the ZLB binds, \(x_t\) is solely determined by equations (10) through (13), given \(x_{t+1}\). The matrices from \(M_1\) through \(M_5\) contain the coefficients that appear in these structural equations. On the other hand, when
the ZLB does not bind, \( x_t \) must be a solution to the above minimization problem subject to equations (11) through (13). The nominal interest rate can then be computed from (10). Matrices \( N_1 \) through \( N_4 \) collect the coefficients in equations (11) through (13).

**Step 2: Value and policy functions for \( t = 8 \)**

Using the algorithm developed by Söderlind (1999), the value and policy functions in period 8 can be computed. The problem in period 8 is to minimize (17) subject to (11) and (13), since we know a priori that the ZLB will never bind after this period. Since \( \omega q_t = 0 \) for every period after period 8, the exogenous state variable, \( s_t \), is also 0 and can be dropped from the argument of the value function. Since the policy function is known to have the solution form \( x_{t+1} = C_{t+1} \hat{K}_{t+1} \), we can substitute this expression into the expectation term in the constraints. Consequently, the optimization problem can be expressed as follows:\(^{24}\)

\[
\hat{K}_t P_t \hat{K}_t = \min_{\{x_t\}} \left\{ \begin{bmatrix} x_T^T, \hat{K}_t \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_t \\ \hat{K}_t \end{bmatrix} + \beta x_t^T R^T P_{t+1} R x_t \right\}
\]

s.t. \((N_2 + N_1 C_{t+1} R) x_t + N_3 \hat{K}_t = 0.\)

After taking the first-order conditions and rearranging, we obtain the following recursive expressions:

\[
x_t = C_t \hat{K}_t,
\]

\[
P_t = C_t^T Q_{11} C_t + 2 C_t^T Q_{12} + Q_{22} + \beta C_t^T R^T P_{t+1} R C_t,
\]

\[
C_t = - (Q_{11} + \beta R^T P_{t+1} R)^{-1} \left\{ Q_{12} + (N_2 + N_1 C_{t+1} R)^T \Gamma_t \right\},
\]

\[
\Gamma_t \equiv \left\{ (N_2 + N_1 C_{t+1} R) (Q_{11} + \beta R^T P_{t+1} R)^{-1} (N_2 + N_1 C_{t+1} R)^T \right\}^{-1} \left\{ N_3 - (N_2 + N_1 C_{t+1} R) (Q_{11} + \beta R^T P_{t+1} R)^{-1} Q_{12} \right\}.
\]

Starting from appropriate initial values for \( P_{t+1} \) and \( C_{t+1} \), we can obtain convergent limits, \( P \) and \( C \), which constitute the value and policy functions in period 8.

**Step 3: Solving backwards**

For periods \( t = 0, 1, 2, ..., 7 \), the problem can be solved backwards given the value and policy functions in period 8. Recall that we solve for all possible zero interest rate combinations over those periods and check whether each combination satisfies the ZLB constraint and the optimality conditions. In our deterministic problem, the policy function is linear and thus can be expressed as \( x_t = X_{k,t} \hat{K}_t + X_{s,t} s_t. \)

\(^{23}C_{t+1} \) is a \( 3 \times 1 \) matrix which contains undetermined coefficients.

\(^{24}\hat{K}_{t+1} = R x_t, \text{ where } R = [0, 0, 1].\)
If the ZLB binds

We substitute $x_{t+1} = x_{k,t+1}K_{t+1} + x_{s,t+1}s_{t+1}$ in the first set of constraints in (53) to obtain the solution:

$$x_{k,t} = - (M_1 x_{k,t+1}R + M_2)^{-1} M_3,$$

$$x_{s,t} = - (M_1 x_{k,t+1}R + M_2)^{-1} (M_1 x_{s,t+1}S^s + M_2 S^q + M_5 S^r),$$

$$S^s = [0_{(8-t) 	imes 1} \ 1_{(8-t)}], \ S^q = [I_2 \ 0_{2x(7-t)}], \ S^r = [0_{1x(8-t)} \ 1].$$

The value function for period $t$ can then be computed by substituting the above results in (52). If we divide the coefficient matrix for the value function into four appropriately-sized blocks of matrices, it can be expressed as

$$P_t = \begin{bmatrix} P_t(11) & P_t(12) \\ P_t(12)^T & P_t(22) \end{bmatrix},$$

where

$$P_t(11) = X_{k,t}^T Q_{11} X_{k,t} + 2X_{k,t}^T Q_{12} + Q_{22} + \beta X_{k,t}^T R P_{t+1}(11) R X_{k,t},$$

$$P_t(12) = X_{k,t}^T Q_{11} X_{s,t} + Q_{12}^T X_{s,t} + X_{k,t}^T Q_{13} S^w + Q_{23} S^w$$

$$+ \beta \left( X_{k,t}^T R P_{t+1}(11) R X_{s,t} + X_{k,t}^T R P_{t+1}(12) S^s \right),$$

$$P_t(22) = X_{s,t}^T Q_{11} X_{s,t} + 2X_{s,t}^T Q_{13} S^w + S^w^T Q_{33} S^w$$

$$+ \beta \left( X_{s,t}^T R P_{t+1}(11) R X_{s,t} + 2X_{s,t}^T R P_{t+1}(12) S^s + S^w^T P_{t+1}(22) S^s \right),$$

and $S^w = [1, 0, \ldots, 0]$ is a $1 \times (8 - t)$ vector.
If the ZLB does not bind

For periods in which the ZLB does not bind, the minimization problem in (52) subject to the second set of constraints in (53) must be solved. By taking the first-order conditions, we obtain the following expressions:

\[ X_{k,t} = - (Q_{11} + \beta R^T P_{t+1}(11) R)^{-1} (Q_{12} + A^T \Gamma_{1,t}), \quad (58) \]

\[ X_{s,t} = - (Q_{11} + \beta R^T P_{t+1}(11) R)^{-1} (Q_{13} S^w + \beta R^T P_{t+1}(12) S^* + A^T \Gamma_{2,t}), \quad (59) \]

\[ A \equiv N_2 + N_1 X_{k,t+1} R, \]

\[ B \equiv N_3, \]

\[ C \equiv N_1 X_{s,t+1} S^* + N_4 S^q, \]

\[ \Gamma_{1,t} \equiv \left\{ A \left( Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} A^T \right\}^{-1} \left\{ B - A \left( Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} Q_{12} \right\}, \]

\[ \Gamma_{2,t} \equiv \left\{ A \left( Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} A^T \right\}^{-1} \left\{ C - A \left( Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} \left( Q_{13} S^w + \beta R^T P_{t+1}(12) S^* \right) \right\}. \]

The coefficient matrix for the value function can be obtained by substituting (58) and (59) into (54) through (57).

Step 4: Consistency with the optimality conditions

For each combination of hitting the ZLB in periods 0 through 7, we compute the corresponding solutions of \( \{x_t\}_{t=0}^7 \) and \( \{\hat{R}_t\}_{t=0}^7 \) using the methods in steps 1 to 3. If \( \hat{R}_t < 1 - 1/\beta \) in any period, that combination should be eliminated because it violates the non-negativity of the nominal interest rate. We also need to check whether the central bank is implementing the zero-interest-rate policy optimally. Given the capital stock in each period and the expectation of economic agents, the central bank must have no incentive to deviate from the zero-interest-rate policy. If, with regard to a particular combination, welfare could be improved if the central bank chose a positive interest rate level instead, such a combination should also be excluded from the set of possible solutions. The combination that survives this elimination process is the unique solution.